9

Oscillatory Motion

Periodic Motion

(*College Physics* 9th ed. pages 437–439/10th ed. pages 445–447)

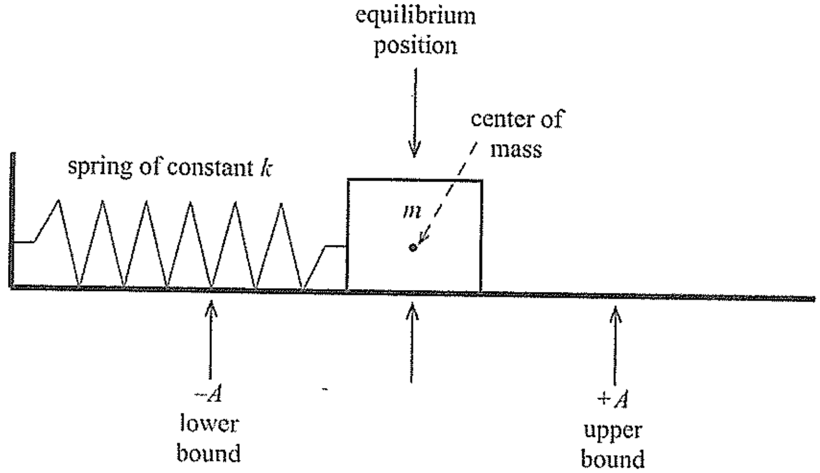
To this point we have studied three types of motion. The simplest is that of a body in translatory equilibrium, a motion consisting of constant speed in an unchanging direction. The second type of motion that is produced by the action of a constant force parallel to the direction of motion is that in which the direction is constant and the speed increases uniformly. Projectile motion is an example of these two types of motion. The third type of motion is uniform circular motion that is produced by a centripetal force of constant magnitude directed inward along the radius of the circular path of the moving body.

There is one common and important type of nonuniformly accelerated motion that can be analyzed rather simply. We call this motion *periodic* or *oscillatory* motion.

Oscillatory motion is quite important in many areas of physics. It is the to-and-fro or vibrating motion of bodies stretched or bent from their normal positions and then released. Such a body moves back and forth along a fixed path, repeating over and over a fixed set of motions and returning to each position and velocity after a definite period of time. Besides being called oscillatory, or periodic, motion, it is also called *harmonic* motion. This motion is caused by varying forces and hence the body experiences varying accelerations.

Consider a block of mass m at rest on a frictionless surface. The block is attached to a spring of elastic constant k that is fastened to a rigid wall. The mass is pulled to the right and is released from rest. The center of mass of the block oscillates between two amplitude positions,

+A and –A. The motion of the oscillating mass is bounded by these two points, +A and –A.



The spring exerts a restoring force on the block that tends to pull or push it back to its initial position, the *equilibrium position*. This force, Hooke's law, is proportional to the displacement x but opposite in direction to the displacement.

$$F = -kx$$

When the block is released, the restoring force, which is the net force, produces an acceleration a:

$$F = ma = -kx$$

$$a = -\frac{k}{m}x$$

The acceleration a is proportional to the displacement x but opposite in direction.

As the block slides toward its equilibrium position, its speed increases but the force, and consequently the acceleration, decreases until it becomes zero when the block reaches the equilibrium position. Because of its momentum, the block continues past the equilibrium position, but at once a retarding force comes into being which increases until the block reaches the amplitude position –A, where it stops momentarily and begins its return trip. At all times during this motion the net force, and hence the acceleration, is proportional to the displacement and directed toward the equilibrium position.

This type of oscillatory motion, when the acceleration is proportional to the displacement and is always directed toward the equilibrium position, is called *simple harmonic motion*, SHM. Simple harmonic motion is always motion along a straight line, the acceleration and velocity constantly changing as the oscillating block moves through its series of positions.

AP Tip

A body can only be in simple harmonic motion when there is a force of restitution.

Amplitude, Period, and Frequency

(*College Physics* 9th ed. pages 438–439/10th ed. pages 446–447)

The *amplitude, A,* of oscillatory motion is the maximum displacement from the equilibrium position. $\pm A$ are the boundaries of the motion. Amplitude is expressed in meters, m.

The *period, T,* of an oscillating body is the time for a complete to-and-fro motion, or a complete oscillation. Period is expressed in seconds, s.

The *frequency, f,* of the oscillatory motion is the number of complete oscillations per second. The frequency is defined as the reciprocal of the period

$$f = \frac{1}{T}$$

Frequency is expressed in hertz, Hz and $1\ \text{Hz} = \frac{1}{\text{s}} = \text{s}^{-1}$.

The *angular frequency,* ω, is defined as

$$\omega = 2\pi f = \frac{2\pi}{T}$$

and it is expressed in $\frac{\text{rad}}{\text{s}}$. It is also expressed in Hz and $1\ \text{Hz} = 1\frac{\text{rad}}{\text{s}} = \text{s}^{-1}$.

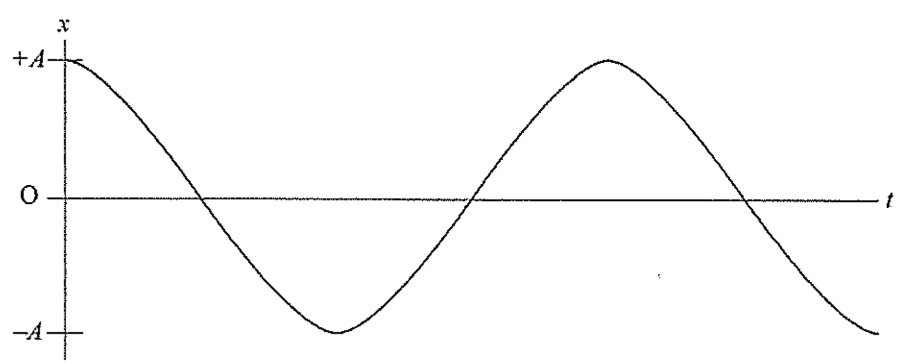
Location of a Harmonic Oscillator

(*College Physics* 9th ed. page 449/10th ed. page 457)

When an oscillator like the block on a frictionless surface that is connected to a spring is pulled to the right, aligning its center of mass to the amplitude position $+A$ and is then released from rest, it undergoes SHM. At any time t after it is released its position x is found by using

$$x = A\cos\omega t$$

Note that the equation is a cosine function. The graph of the cosine function starts at $+A$ and oscillates through $-A$ back to $+A$ for a complete cycle or oscillation.



If the oscillator is given a shove left or right at $+A$ to begin the motion the situation becomes complex and will not be handled here.

THE ACCELERATION OF A HARMONIC OSCILLATOR

(*College Physics* 9th ed. pages 449–450/10th ed. pages 457–458)

Above you saw that the acceleration of the oscillator is a function of position, $a = f(x)$ or $a = -\frac{k}{m}x$. The acceleration of the oscillator is also given by

$$a = -\omega^2 x$$

The maximum acceleration, a_{max}, occurs at the amplitude positions, $a_{max} \propto \pm A$ and is expressed as

$$a_{max} = -\omega^2 A$$

Since $a = -\frac{k}{m}x$ and $a = -\omega^2 x$ then $\omega^2 = \frac{k}{m}$ and taking the square root of both sides yields

$$\omega = \sqrt{\frac{k}{m}}$$

Frequency can now be expressed as

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

And the period is

$$T = 2\pi\sqrt{\frac{m}{k}}$$

SAMPLE PROBLEM 1

A 0.40 kg mass is attached to a vertical spring of $k = 10.0\ \text{N/m}$. The mass is displaced 0.04 m vertically downward from equilibrium and is released from rest. The mass executes SHM. Neglecting friction and air resistance, find

(a) the angular frequency

(b) the period of motion

(c) the maximum acceleration experienced by the mass

SOLUTION TO PROBLEM 1

(a) The angular frequency is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{10.0\ \text{N/m}}{0.40\ \text{m}}} = \sqrt{25\ \text{1/s}^2} = 5.0\ \frac{\textbf{rad}}{\textbf{s}}$$

(b) The period of the oscillator is

$$T = \frac{2\pi}{\omega} = \frac{2\pi \text{ rad}}{5.0 \text{ rad/s}} = \mathbf{1.26\ s}$$

(c) The maximum acceleration is

$$a_{max} = -\omega^2 A = -\left(25.0 \text{ rad}^2/\text{s}^2\right)(0.04 \text{ m}) = \mathbf{-1.00\ \frac{m}{s^2}}$$

THE ENERGY OF A HARMONIC OSCILLATOR

(*College Physics* 9th ed. pages 441–445/10th ed. pages 449–453)

The total energy, E, of a mechanical system is the sum of its kinetic energy, K, and potential energy, U, or $E = K + U$.

The kinetic energy is zero at the amplitude positions since the body is momentarily at rest. The kinetic energy is always zero at $\pm A$. At the equilibrium position, $x = 0$, the kinetic energy of the body is a maximum and is $K = \frac{1}{2}mv^2$.

The elastic potential energy, $U = \frac{1}{2}kx^2$, of the system is zero at the equilibrium position since $x = 0$. At the amplitude positions U is a maximum.

The total energy of a simple harmonic oscillator is

$$E = \frac{1}{2}kA^2$$

Without proof, the velocity of the oscillator at any position, x, is given by

$$v = \pm\omega\sqrt{A^2 - x^2}$$

Therefore

$$v_{max} = \pm\vec{\omega}A$$

SAMPLE PROBLEM 2

A 0.50 kg block is connected to a horizontal spring, $k = 20.0$ N/m. The block is pulled to the right and then released from rest where it begins to oscillate on a horizontal, frictionless surface with amplitude 0.03 m. Analyze the motion of the block.

SOLUTION TO PROBLEM 2

The spring exerts a force on the block and that force, Hooke's law, is a force of restitution making the motion simple harmonic.

The acceleration is zero at equilibrium. Maximum acceleration occurs at $\pm A$.

$$a_{max} = \pm\omega^2 A = \pm\frac{k}{m} \text{ since } \omega = \sqrt{\frac{k}{m}}$$

$$a_{max} = \pm\left(\frac{20\ \text{N}/\text{m}}{0.5\ \text{kg}}\right)(0.03\ \text{m}) = \pm\mathbf{1.20\ \frac{m}{s^2}}$$

Maximum velocity occurs at equilibrium where $x = 0$.

$$v_{max} = \pm\omega A = \pm\sqrt{\frac{k}{m}}A = \pm\sqrt{\frac{20.0\ \text{N}/\text{m}}{0.5\ \text{kg}}}(0.03\ \text{m}) = \pm\mathbf{0.19\ \frac{m}{s}}$$

The total energy of the system is found by

$$E = \frac{1}{2}kA^2 = \frac{1}{2}(20.0\ \text{N}/\text{m})(0.03\ \text{m})^2 = \mathbf{0.009\ J}$$

The angular frequency is related to frequency of vibration or oscillation by

$$\omega = 2\pi f \text{ and then } f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{20.0\ \text{N}/\text{m}}{0.5\ \text{kg}}} = \mathbf{1.007\ Hz}$$

Period is the reciprocal of the frequency of vibration.

$$T = \frac{1}{f} = \frac{1}{1.007\ \text{s}^{-1}} = \mathbf{0.993\ s}$$

The angular frequency $\omega = \sqrt{\frac{k}{m}}$ will allow us to write the position equation of the oscillator.

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{20.0\ \text{N}/\text{m}}{0.5\ \text{kg}}} = \mathbf{6.325\ \frac{rad}{s}}$$

Since the equation of motion is $x = A\cos\omega t$, then $\mathbf{x = 0.03\cos(6.325t)}$

SAMPLE PROBLEM 3

Consider the system analysis above.

(a) Determine the velocity of the block when its displacement is 0.02 m and it is moving to the right.

(b) Find the acceleration of the block when its displacement is 0.02 m and it is moving to the right.

(c) What are the kinetic energy, elastic potential energy, and total energy of the oscillator when it is at the 0.02 m position to the right of equilibrium?

SOLUTION TO PROBLEM 3

(a) $v = \pm\omega\sqrt{A^2 - x^2} = \pm(6.325\ \text{s}^{-1})\sqrt{(0.03\ \text{m})^2 - (0.02\ \text{m})^2} = \pm 0.141\ \frac{\text{m}}{\text{s}}$

Since the block travels to the right, the velocity is positive and $v = \mathbf{0.141\ \frac{m}{s}}$

(b) $a = \pm\omega^2 x = \pm\left(6.325\ \text{s}^{-1}\right)^2 (0.02\ \text{m}) = \pm 0.800\ \frac{\text{m}}{\text{s}^2}$

Since the block moves right, the elastic force acts in the opposite direction, slowing it. Thus $a = \mathbf{-0.800\ \frac{m}{s^2}}$.

(c) Kinetic energy: $K = \frac{1}{2}mv^2 = \frac{1}{2}(0.50\ \text{kg})\left(0.141\ \frac{\text{m}}{\text{s}}\right)^2 = \mathbf{0.005\ J}$

Potential energy: $U = \frac{1}{2}kx^2 = \frac{1}{2}\left(20.0\ \text{N}/\text{m}\right)(0.02\ \text{m})^2 = \mathbf{0.004\ J}$

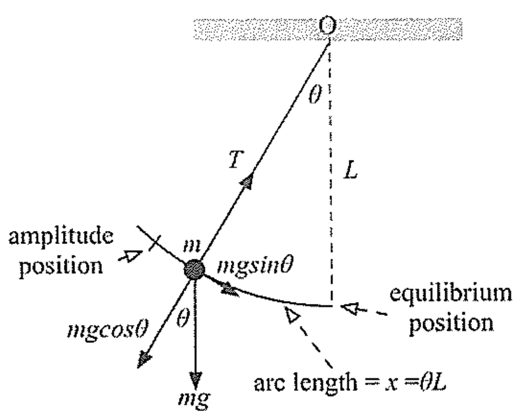
Total energy: $E = K + U = 0.005\ \text{J} + 0.004\ \text{J} = \mathbf{0.009\ J}$

THE SIMPLE PENDULUM

(*College Physics* 9th ed. pages 451–454/10th ed. pages 460–462)

A simple pendulum is an idealized body that consists of a light, inextensible cord, one end of which is attached to a fixed support O, and a small mass, called a *pendulum bob*, which is attached to the other end. When at rest the bob hangs at the equilibrium position. When pulled aside to the amplitude position and released from rest, it travels an arc length x through the equilibrium position to the amplitude position on the other side. If the arc length is made very small so that it approximates a straight line, the motion of the pendulum is simple harmonic.

Simple Pendulum



The forces acting on the bob are its weight, mg, and the tension, T, in the cord. Resolve the weight vector, mg, into a radial component, $mg\cos\theta$, and a tangential component, $mg\sin\theta$. The radial component along with the tension T supply the centripetal acceleration that keeps the bob moving along a circular arc. The tangential component is the restoring force acting on the bob tending to return it to the equilibrium position. The force of restitution is:

$$F = -mg\sin\theta$$

Notice that the force of restitution is proportional to $\sin\theta$ and not the angular displacement, θ. For very small angles, $1° < \theta < 6°$, the sine of the angle in degrees is about equal to the angle expressed in radians. See the table that follows. The displacement along the arc is $x = \theta L$, and for small angles this is nearly straight-line motion.

The relationship between degrees and radians can be written as $1° \times \frac{\pi \text{ rad}}{180°}$.

Define deviation as $\Delta = |\sin\theta - \theta(\text{ in radians})|$

Define percent deviation as $\%\Delta = \frac{\Delta \times 100\%}{\sin\theta}$

Degrees and Radians Table for Small Angles

angle θ	$\sin\theta$	θ in radians	deviation Δ	% deviation
1°	0.01745	0.01745	0	0
2°	0.03490	0.03491	0.00001	0.03%
3°	0.05234	0.05236	0.00002	0.04%
4°	0.06976	0.06981	0.00005	0.06%
5°	0.08716	0.08727	0.00011	0.13%
10°	0.17365	0.17453	0.00088	0.50%

The arc length, x, is related to the length, L, of the pendulum by $x = \theta L$ and from this $\theta = \frac{x}{L}$. For very small angles $\sin\theta \approx \theta$, we can write

$$F = -mg\sin\theta = -mg\theta = -mg\frac{x}{L}$$

Rearranging terms yields $F = -\left(\frac{mg}{L}\right)x$ which is a form of Hooke's law, $F = -kx$.

Define $k = \frac{mg}{L}$. For an oscillating system, $T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{mg/L}}$.

The period for a simple pendulum is

$$T = 2\pi\sqrt{\frac{L}{g}}$$

AP Tip

The period of the simple pendulum is independent of the mass of the pendulum bob.

SAMPLE PROBLEM 4

Calculate the length of a simple pendulum with a period of 4.00 seconds.

SOLUTION TO PROBLEM 4

By definition $T = 2\pi\sqrt{\frac{L}{g}}$ and solving for the length yields

$$L = \frac{T^2 g}{4\pi^2} = \frac{(4.00\text{ s})^2\left(9.80\ \text{m}/\text{s}^2\right)}{4\pi^2} = \mathbf{3.97\ m}$$

SAMPLE PROBLEM 5

Find an expression for the angular frequency of a simple pendulum.

SOLUTION TO PROBLEM 5

Period is $T = 2\pi\sqrt{\frac{L}{g}}$ and frequency is defined as $f = \frac{1}{T}$.

For the simple pendulum

$f = \frac{1}{2\pi\sqrt{\frac{L}{g}}} = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$. Cross-multiplying yields $2\pi f = \sqrt{\frac{g}{L}}$.

Angular frequency is defined by $\omega = 2\pi f$, the angular frequency of the simple pendulum is then $\omega = \sqrt{\frac{g}{L}}$.

OSCILLATORY MOTION: STUDENT OBJECTIVES FOR THE AP EXAM

- You should understand the relationship between the angular frequency, the ordinary frequency, and the period of an oscillator.
- You should be able to define the term simple harmonic motion.
- You should be able to explain the relationship between the kinetic, potential, and total energies for a harmonic oscillator.
- You should be able to explain the source of the restoring force for the simple pendulum.
- You should be able to analyze the motion of a system in simple harmonic motion.