8

GRAVITATION

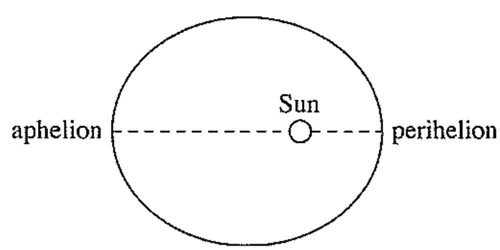
KEPLER'S LAWS OF PLANETARY MOTION

(*College Physics* 9th ed. pages 221–224/10th ed. pages 226–229)

Over a three-year span from 1599 to 1601, Johannes Kepler (1571–1630) worked with the noted Danish astronomer Tycho Brahe (1546–1601). Brahe had amassed over twenty-five years of carefully measured observations of the planets, moon, and stars. Keep in mind that the telescope was invented in 1608. The quality and accuracy of the observations led Kepler to develop his three laws of planetary motion.

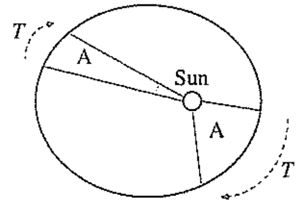
Kepler's first law (1609): *The planets move in elliptical orbits with the Sun at one of the foci.*

At its most fundamental, the first law shows that planets trace out elliptical orbits with the Sun at one of the focal points. Nothing occupies the other focal point. The *eccentricities* of the planetary orbits are so small that they can almost be considered circles.



When a planet is at its most distant point from the Sun, the position is called *aphelion*. At its closest approach the planet is at *perihelion*. Helion comes from the ancient Greek and it means Sun. For Earth orbit, it is *apogee* and *perigee*. For moon or lunar orbit it is *apolune* and *perilune*.

Kepler's second law (1609): *The straight line joining the Sun and any planet sweeps out equal areas, A, in equal intervals of time, T.*



The first law gives all possible positions of a planet, but no time indications. The second law implies that the velocity of the planet in its orbit decreases with increasing distance to the Sun. A planet moves faster when at *perihelion*, and slower at *aphelion*. The second law gives the time dependence critical to predicting planetary positions.

Kepler's third law (1619): *The ratio of the square of the period, T, of one full orbit about the Sun and the cube of the radius, R, of the orbit is a constant.*

Mathematically, the third law can be expressed as $\frac{T^2}{R^3} = \text{constant}$. The constant can be deduced from other considerations and then

$$\frac{T^2}{R^3} = \frac{4\pi^2}{GM}$$

G is a constant we will encounter in the next section. M is the mass of the central body and T is the period or time of one complete orbit.

NEWTON'S UNIVERSAL LAW OF GRAVITATION

(*College Physics* 9th ed. pages 214–217/10th ed. pages 219–222)

It is often said incorrectly that Newton discovered gravity. What Newton actually discovered was the universal law of gravitation.

Basically, he wrote, *every particle in the universe attracts every other particle with a force that is directly proportional to their masses and is inversely proportional to the inverse square of the distance between them*, or

$$\vec{F} \propto \frac{m_1 m_2}{R^2}$$

To make the proportionality an equation we introduce the constant, G, and call it the *universal gravitational constant*. The universal law of gravitation is then written:

$$\vec{F} = G\frac{m_1 m_2}{R^2}$$

The relationship is called an *inverse square law* since the force drops off as the inverse of the square of the distance, $1/R^2$, between interacting bodies, m_1 and m_2. It was not until 1798 when the English experimentalist Henry Cavendish (1731–1810) first measured the universal gravitational constant, G. In terms of today's values it is

$$G = 6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$$

Gravity is one of the four fundamental forces in nature. It is the weakest of the four. Gravity is the primary force acting on astronomical bodies. It is always an attractive force. The force of gravity between two ordinary bodies on the Earth is negligibly small.

There is much about gravity we do not completely understand. All bodies of mass m generate a gravitational field, a **g**-field. A small mass generates a tiny field and an enormous body, such as a planet, generates an enormous field. It is the interaction of these **g**-fields that is the gravitational force of attraction. All bodies have a center of mass, c.m. A symmetrical body such as a sphere has its c.m. at its very center. All bodies behave as if all their mass were concentrated at the c.m. The **g**-field converges into the center of mass. The **g**-field intensity, $\vec{g}$, is a vector. It has magnitude and direction.

Sample Problem 1

Spent uranium is a very dense material. A cubic foot of it would weigh over 1200 pounds. What gravitational attraction would exist between two identical 100 kg spheres of spent uranium? The spheres are positioned so that their centers of mass are 3.0 m apart.

Solution to Problem 1

The gravitational force is

$$F = \frac{Gm_1m_2}{R^2} = G\left(\frac{m}{R}\right)^2$$

$$= \left(6.67\times10^{-11}\ \mathrm{N\cdot m^2/kg^2}\right)\left(\frac{100\ \mathrm{kg}}{3.0\ \mathrm{m}}\right)^2$$

$$= \mathbf{7.4\times10^{-8}\ N}$$

This gravitational force is on the order of the force an amoeba generates pushing off the surface of a tiny grain of sand in a few drops of water.

The Acceleration Due to Gravity

(*College Physics* 9th ed. pages 216–217/10th ed. pages 221–222)

The gravitational force acting on a body is its weight, $\vec{w} = m\vec{g}$. Since weight is gravitational force we can write

$$F = w = G\frac{mM}{R^2} = m\left(\frac{GM}{R^2}\right)$$

Assuming any planet or moon as being spherical and of uniform or homogenous composition, the acceleration due to gravity at the surface is then

$$\vec{g}_P = G\frac{M_P}{R_P^2}$$

$\vec{g}_p$ is the acceleration due to gravity on the surface of the planet, M_p the mass of the planet and R_p the planet radius. Think of it as the distance from the c.m. of the planet to the surface.

Sample Problem 2

The planet Mars has a mass of 6.387×10^{23} kg and a radius of 3.332×10^{6} m. To two decimal places, determine the acceleration due to gravity for the surface of Mars assuming that it is spherical and homogeneous. The mass of Mars is 6.386×10^{24} kg and its radius is 3.332×10^{6} m.

Solution to Problem 2

For the surface of Mars:

$$g_{Mars} = G\frac{M_{Mars}}{R^2_{Mars}}$$
$$= \frac{\left(6.673 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}\right)\left(6.387 \times 10^{23}\ \mathrm{kg}\right)}{\left(3.332 \times 10^{6}\ \mathrm{m}\right)^2}$$
$$= \mathbf{3.83\ m/s^2}$$

Mars is considerably less massive and has a smaller radius than the Earth. At its surface it has 39% of the acceleration due to gravity as the Earth.

Sample Problem 3

On the surface of the Earth a 100 kg body has a weight $w = mg = (100\ \mathrm{kg})\left(9.8\ \mathrm{m/s^2}\right) = 980\ \mathrm{N}$. What is the weight of this body on the surface of Mars?

Solution to Problem 3

From the above problem $g_{Mars} = 3.83\ \mathrm{m/s^2}$ and the weight on the surface of Mars is

$$w_{Mars} = mg_{Mars} = (100\ \mathrm{kg})\left(3.83\ \mathrm{m/s^2}\right) = \mathbf{383\ N}$$

Sample Problem 4

Assume that the moon moves about the Earth in circular orbit having a radius of 3.846×10^{8} m. The period of revolution of the moon is 27.32 days.

(a) What is the centripetal acceleration of the moon in its orbit?

(b) Knowing the Earth has a mass of 5.97×10^{24} kg, what is the acceleration due to gravity at any point along the orbit of the moon? Express the answer as a decimal value.

Solution to Problem 4

(a) Convert the period, T, into seconds.

$$27.32\ \mathrm{day} \times \frac{8.64 \times 10^{4}\ \mathrm{s}}{1\ \mathrm{day}} = 2.36 \times 10^{6}\ \mathrm{s}$$

Next find the orbital speed of the moon:

$$v = \frac{C}{T} = \frac{2\pi R}{T} = \frac{2\pi\left(3.846\times10^{8}\ \text{m}\right)}{\left(2.36\times10^{6}\ \text{s}\right)} = 1.02\times10^{3}\ \text{m/s}$$

The centripetal acceleration experienced by the moon is $a_c = \frac{v^2}{R}$

and $a_c = \frac{\left(1.02\times10^{3}\ \text{m/s}\right)^2}{\left(3.846\times10^{8}\ \text{m}\right)} = \mathbf{0.0027\ m/s^2}$

(b) The acceleration due to gravity at a point 3.846×10^{8} m from the center of the Earth:

$$\begin{aligned} g &= G\frac{M_{Earth}}{R^2} \\ &= \frac{\left(6.73\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{s}^2\right)\left(5.97\times10^{24}\ \text{kg}\right)}{\left(3.846\times10^{8}\ \text{m}\right)^2} \\ &= \mathbf{0.0027\ m/s^2} \end{aligned}$$

In Newton's earliest investigation into gravitation he suspected that gravity was an inverse square law. In terms of the SI, which did not exist at the time of Newton, he knew the distance between the moon and Earth was 60 Earth radii. He reasoned, as we did above, that the centripetal acceleration at the moon's orbit was $0.0027\ \text{m/s}^2$. Then taking g for the surface of the Earth and dividing by the square

$$\frac{9.8\ \text{m/s}^2}{60^2} = \frac{9.8\ \text{m/s}^2}{3600} = 0.0027\ \text{m/s}^2$$

This was all the proof Newton needed to know he was correct. Universal gravitation was an inverse square law.

CIRCULAR ORBIT SPEED

(*College Physics* 9th ed. pages 217–220/10th ed. pages 222–225)

When a space vehicle is launched atop a rocket from the surface of the Earth to orbit, the initial liftoff direction is vertically upward. As the rocket gains altitude, control jets and fins slowly make it turn toward a horizontal trajectory. At the proper point, the satellite separates from the rocket. With too low of an initial speed, the vehicle will follow a nearly parabolic trajectory and will strike the Earth. With just the right speed, the satellite will follow a circular orbit of radius R. We call this speed *the circular orbit speed*. At higher speed the satellite will go into an elliptical orbit or will completely escape the Earth.

A satellite in a circular orbit about a central body of mass M and orbiting a distance R from the center of mass of the body will have an orbital speed given by

$$v = \sqrt{\frac{GM}{R}}$$

The equation holds for any satellite moving in a circular orbit around any astronomical body. The general equations for a satellite moving in elliptical orbits are rather complicated and will not be considered here.

SAMPLE PROBLEM 5

A 500 kg satellite is placed into a circular orbit 300 km above the surface of the Earth.

(a) What is its orbital speed?

(b) What is its orbital period, T?

SOLUTION TO PROBLEM 5

(a) The orbital speed is independent of the mass of the orbiting satellite. The radius of the orbit, R, is the distance from the center of the c.m. of the Earth to the c.m. of the satellite:

$$R = R_E + h = 6{,}378 \text{ km} + 300 \text{ km} = 6{,}678 \text{ km} \times \frac{1{,}000 \text{ m}}{1 \text{ km}} = 6.678 \times 10^6 \text{ m}$$

$$v_{orbit} = \sqrt{\frac{GM_{Earth}}{R}}$$

$$= \sqrt{\frac{\left(6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2\right)\left(5.97 \times 10^{24} \text{ kg}\right)}{\left(6.678 \times 10^6 \text{ m}\right)}}$$

$$= \mathbf{7.72 \times 10^3 \ m/s}$$

(b) By definition, speed is defined as $v = \frac{s}{T}$. The distance traveled in one orbit is $s = C = 2\pi R$. The period then is

$$T = \frac{2\pi R}{v} = \frac{2\pi\left(6.678 \times 10^6 \text{ m}\right)}{7.72 \times 10^3 \text{ m/s}} = \mathbf{5.43 \times 10^3 \ s = 90.6 \ min}$$

(b) Alternative solution: Using Kepler's third law $\frac{T^2}{R^3}=\frac{4\pi^2}{GM}$ and solving for T

$$T=\sqrt{\frac{4\pi^2R^3}{GM_E}}=2\pi R\sqrt{\frac{R}{GM}}$$

$$=2\pi\left(6.678\times10^6\ \text{m}\right)\sqrt{\frac{\left(6.678\times10^6\ \text{m}\right)}{G\left(5.97\times10^{24}\ \text{kg}\right)}}$$

$$=\mathbf{5.43\times10^3\ s=90.6\ min}$$

Note that in (b) and (b) alternative solutions both use the same equation but were approached differently.

Sample Problem 6

A weather satellite circles the Earth in a *geosynchronous orbit* with a period of exactly 1 day. In this way the satellite is always over the same spot all the time. Find the altitude of a geosynchronous orbit.

Solution to Problem 6

A satellite moving in a circular orbit of radius R covers a distance of $2\pi R$ in a time period T. First, convert 1 day into seconds, $1\ \text{d}=(24\ \text{h})(60\ \text{min/1 h})(60\ \text{s/1 min})=8.64\times10^4\ \text{s}$. Next write Kepler's third law $T^2=\left(\frac{4\pi^2}{GM_E}\right)R^3$. Solving for R gives

$$R=\sqrt[3]{\frac{GMT^2}{4\pi^2}}$$

$$=\sqrt[3]{\frac{\left(6.672\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2\right)\left(5.97\times10^{24}\ \text{kg}\right)\left(8.64\times10^4\ \text{s}\right)^2}{4\pi^2}}$$

$$R=4.23\times10^7\ \text{m}$$

Next we subtract the radius of the Earth to find the altitude, h.

$$h=4.23\times10^7\ \text{m}-0.64\times10^7\ \text{m}=\mathbf{3.59\times10^7\ m}$$

The satellite orbits at 22,310 miles above the surface of the Earth.

Escape Velocity

(*College Physics* 9th ed. page 220/10th ed. page 225)

For a space vehicle to escape from the gravitational field of the Earth and never return, it must be launched with a velocity greater than that required to place it into Earth's orbit. The minimum *escape velocity* from the surface of any planet or moon is

$$v_{escape}=\sqrt{\frac{2GM}{R}}$$

The escape velocity of a space vehicle is independent of the mass m of the vehicle.

SAMPLE PROBLEM 7

Find the escape velocity from the surface of the Earth required to send a 1000 kg space probe into the depths of outer space.

SOLUTION TO PROBLEM 7

Escape velocity is independent of the mass of the space probe being launched. The escape velocity is

$$v_{escape} = \sqrt{\frac{2GM_{Earth}}{R_{Earth}}}$$

$$= \sqrt{2\frac{\left(6.67\times10^{-11}\,\mathrm{N\cdot m^2/kg^2}\right)\left(5.97\times10^{24}\,\mathrm{kg}\right)}{\left(6.378\times10^{6}\mathrm{m}\right)}}$$

$$= \mathbf{1.12\times10^{4}\ m/s}$$

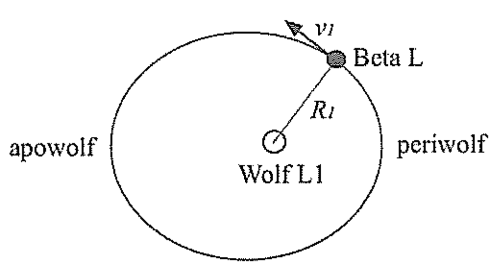
The above equation can be modified for the Earth, and the escape velocity from the Earth's surface takes the form

$$v = \sqrt{2gR_E}$$

ANGULAR MOMENTUM

(*College Physics* 9th ed. page 222/10th ed. page 227)

Planets move in curved paths and therefore have angular momentum $\vec{L} = M_{planet}\vec{v}R$. When the net force acting on the system is zero, there is no change in angular momentum, L, and angular momentum is conserved. As a planet gets closer to the Sun it must orbit faster in order to maintain constant angular momentum. The mass of the planets is constant. So the law of conservation of angular momentum requires that the product vR remains constant. This is another way of expressing Kepler's second law.



SAMPLE PROBLEM 8

An Earth-size planet named Beta L circles the star Wolf L1. When Beta L is at a distance $R_1 = 1.50\times10^{11}$ m from Wolf L1, it is observed to have an orbital speed of 30.0×10^{3} m/s. What is the orbital speed of Beta L at periwolf ($R_p = 1.00\times10^{10}$ m) and apowolf (3.40×10^{11} m)?

SOLUTION TO PROBLEM 8

The law of conservation for the planet is $\Sigma L_0 = \Sigma L$ and $M_{planet} v_1 R_1 = M_{planet} v_{peri} R_{peri}$. Note the mass of the planet appears on both sides of the equation and will divide out. To find the orbital speed at periwolf we now write

$$v_{peri} = \frac{v_1 R_1}{R_{peri}} = \frac{\left(30.0 \times 10^3 \ \mathrm{m/s}\right)\left(1.5 \times 10^{11} \ \mathrm{m}\right)}{\left(1.00 \times 10^{10} \ \mathrm{m}\right)} = \mathbf{450 \times 10^3 \ m/s}$$

For apowolf

$$v_{apo} = \frac{v_1 R_1}{R_{apo}} = \frac{\left(30.0 \times 10^3 \ \mathrm{m/s}\right)\left(1.5 \times 10^{11} \ \mathrm{m}\right)}{\left(3.4 \times 10^{11} \ \mathrm{m}\right)} = \mathbf{13.2 \times 10^3 \ m/s}$$

GRAVITATIONAL POTENTIAL ENERGY

(*College Physics* 9th ed. pages 219–219/10th ed. pages 222–224)

Since gravity is a conservative force, we can define a potential energy associated with it. Recall that the work done, *W*, in lifting a mass, *m*, from one point to another in the **g**-field of the Earth equals the gain in the potential energy, *U*. Work is only done against gravity when the displacement is radial. Moving a body sideways requires no work. There is no vertical displacement.

Recall that the word arbitrary is associated with a choice. We must choose where we start and where we finish. If we choose $U = 0$ at $R = \infty$, then the *gravitational potential energy* is

$$U = -\frac{GmM}{R}$$

SAMPLE PROBLEM 9

A 2,000 kg satellite that is 0.23×10^6 m above the surface of the Earth is in a circular orbit. Calculate the potential energy of the satellite.

SOLUTION TO PROBLEM 9

We need the distance from the center of the Earth to the satellite.

$$R = h + R_E = 0.23 \times 10^6 \ \mathrm{m} + 6.4 \times 10^6 \ \mathrm{m} = 6.63 \times 10^6 \ \mathrm{m}$$

Next, we define the potential energy

$$U = -\frac{GmM}{R}$$

$$= -\frac{\left(6.672 \times 10^{-11} \ \mathrm{N \cdot m^2/kg^2}\right)\left(2 \times 10^3 \ \mathrm{kg}\right)\left(5.97 \times 10^{24} \ \mathrm{kg}\right)}{6.63 \times 10^6 \ \mathrm{m}}$$

$$U = \mathbf{-1.21 \times 10^{11} \ J}$$

WEIGHTLESSNESS

(*College Physics* 9th ed. page 210/10th ed. page 215)

Astronauts orbiting the Earth in an artificial satellite feel *apparent weightlessness*. This is similar to the sensation experienced by a person in a freely falling elevator. When the elevator accelerates downward with uniform acceleration *a*, the apparent weight of the person in the elevator is $w = m(g - a)$. When the elevator accelerates downward, the passenger feels lighter.

If the elevator is in *free fall*, the downward acceleration is $a = g$, and $w = m(g - g) = 0$. Thus, a person feels apparent weightlessness in a freely falling elevator.

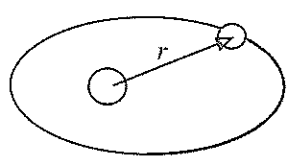
A satellite is in a continuous state of free fall. Although the force of gravity acts on the satellite, an astronaut inside experiences apparent weightlessness.

GRAVITATION: STUDENT OBJECTIVES FOR THE AP EXAM

- You should be able to explain the differences between mass and weight.
- You should be able to state Newton's universal law of gravitation.
- You should be able to calculate gravitational forces between two bodies.
- You should be able to determine the orbital speed and angular momentum of a satellite in a circular orbit.
- You should understand the concept of weightlessness.
- You should be able to explain why the moon does not crash into the Earth despite the large gravitational force acting on it.

MULTIPLE-CHOICE QUESTIONS

1. A radius vector, *r*, is a straight line that runs from the center of mass of one body to the center of mass of a second body, as in the diagram below depicting the Earth and a satellite. The radius vector changes length as the satellite moves toward perigee (closest distance to the Earth) because of gravitational forces between the Earth and the satellite.



The velocity of the center of mass as the satellite approaches perigee will
(A) increase
(B) decrease
(C) remain constant
(D) be zero