7

CIRCULAR MOTION AND ROTATIONAL MOTION

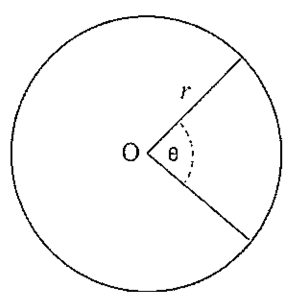
ANGULAR MOTION

(*College Physics* 9th ed. pages 198–199/10th ed. page 203)

In our everyday world, angles are commonly measured and expressed in degrees, where one full turn or one complete *revolution* is 360°. In the study of physics, a more suitable unit, the *radian*, rad, is used.

$$\text{angle in radians} = \frac{\text{arc length}}{\text{radius}}$$

$$\theta = \frac{s}{r}$$



Since the circumference of a circle of radius r is $2\pi r$, there are 2π rad in one complete revolution, rev.

$$1 \text{ rev} = 360° = 2\pi \text{ rad}$$

and

$$1 \text{ rad} = 57.3°$$

AP Tip

All counterclockwise (CCW) rotations are considered positive and all clockwise (CW) rotations are negative.

Angular Velocity

(*College Physics* 9th ed. pages 199–200/10th ed. pages 204–205)

The *angular velocity,* ω, defines how rapidly a body is turning or spinning or rotating about an axis.

$$\text{angular velocity} = \frac{\text{angular displacement}}{\text{time}}$$

$$\omega = \frac{\theta}{t}$$

Angular velocity, ω, is expressed in radians per second, rad/s.

$$1\ \frac{\text{rev}}{\text{s}} = 2\pi \frac{\text{rad}}{\text{s}}$$

The linear velocity, v, of a particle that travels in a circle of radius r with uniform angular velocity, ω, is related by

linear velocity = angular velocity × radius of the circle

or

$$v = \omega r$$

The time required to make one complete revolution is the period, T, and

$$T = \frac{2\pi}{\omega}$$

Since $f = \frac{1}{T}$, the angular frequency, ω, is $\omega = 2\pi f$.

Angular Acceleration

(*College Physics* 9th ed. page 201/10th ed. pages 205–206)

A rotating body whose angular velocity changes from ω_0 to ω in a time interval t undergoes an *angular acceleration,* α.

$$\text{angular acceleration} = \frac{\text{change in angular velocity}}{\text{time interval}}$$

or

$$\alpha = \frac{\omega - \omega_0}{t}$$

The unit for angular acceleration is the rad/s^2. If the angular velocity increases, ω and α have the same sign. If the angular velocity decreases, ω and α will have opposite signs.

In the study of linear motion we found that when a body undergoes uniform acceleration over a time interval, its final velocity is $v = v_0 + at$. In rotational motion, the final angular velocity for a rotating body will be

$$\omega = \omega_0 + \alpha t$$

The linear displacement of an accelerating body is $x = v_0 t + \frac{1}{2}at^2$. The angular displacement is found by

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

The time independent kinematic equation is $v^2 = v_0^2 + 2ax$ and its counterpart in rotational kinematics is

$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

Sample Problem 1

A wheel of radius $r = 0.30$ m spins at the rate of 900 rpm.

(a) What is the angular velocity of all points on the wheel?

(b) If the wheel slows uniformly to 60 rpm in 15 s, what angular acceleration does the wheel experience?

Solution to Problem 1

(a) 900 rpm = 900 revolutions per minute. To find the angular velocity we need to convert revolutions into radians and minutes into seconds.

$$\omega = 900\ \frac{\text{rev}}{\text{min}} \times \frac{2\pi\ \text{rad}}{1\ \text{rev}} \times \frac{1\ \text{min}}{60\ \text{s}} = \mathbf{94.2\ rad/s}$$

(b) First we need to find the final angular and every point on the wheel has the same angular velocity:

$$\omega = 60\ \frac{\text{rev}}{\text{min}} \times \frac{2\pi\ \text{rad}}{1\ \text{rev}} \times \frac{1\ \text{min}}{60\ \text{s}} = 6.28\ \text{rad/s}.$$

Define angular acceleration and set the initial angular velocity as ω_0.

$$\alpha = \frac{\omega - \omega_0}{t} = \frac{6.28\ \text{rad/s} - 94.2\ \text{rad/s}}{15\ \text{s}} = \mathbf{-5.9\ rad/s^2}$$

Sample Problem 2

A small pulley attached to the shaft of an electric motor has a radius $r = 0.05$ m and is turning at $\omega_0 = 5$ rad/s and speeds up to $\omega = 8$ rad/s in 2.5 s.

(a) What acceleration does the pulley experience?

(b) What is the angular displacement during this time period?

(c) How many revolutions, n, is this?

SOLUTION TO PROBLEM 2

(a) Both the initial and final angular velocities are in rad/s. Then

$$\alpha = \frac{\omega - \omega_0}{t} = \frac{8\ \text{rad}/\text{s} - 5\ \text{rad}/\text{s}}{2.5\ \text{s}} = \mathbf{1.2\ rad/s^2}$$

(b) Since we know the time period,

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 = \left(5\ \frac{\text{rad}}{\text{s}}\right)(2.5\ \text{s}) + \frac{1}{2}\left(1.2\ \frac{\text{rad}}{\text{s}^2}\right)(2.5\ \text{s})^2 = \mathbf{16.2\ rad}$$

(c) To find the number of revolutions, n, we use a conversion factor:

$$n = 16.2\ \text{rad} \times \frac{1\ \text{rev}}{2\pi\ \text{rad}} = \mathbf{2.6\ rev}$$

ANGULAR AND TANGENTIAL RELATIONSHIPS

(*College Physics* 9th ed. pages 203–206/10th ed. pages 208–211)

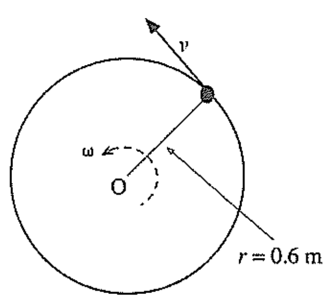
When a wheel of radius, r, rotates about its axis, a point on the rim of the wheel can be described in terms of the *circumferential* distance, s, it has moved, its tangential speed, v, and its tangential acceleration, a. These quantities are related to the angular displacement, θ, angular velocity, ω, and the angular acceleration, α, by the following relationships:

$$s = \theta r \qquad v = \omega r \qquad a = \alpha r$$

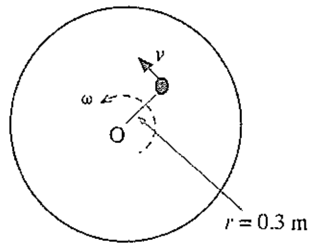
SAMPLE PROBLEM 3

A bicycle wheel mounted on a test frame has a diameter of 1.2 m and spins at the rate of 4.0 rad/s.

(a) What is the linear or tangent speed of a particle on the circumference of the wheel?



(b) What is the linear or tangent speed of a particle 0.3 m from the center of the wheel?



SOLUTION TO PROBLEM 3

(a) The relationship between the angular velocity of a wheel and the linear velocity at any point on the wheel is

$$v = \omega r = \left(4.0 \ \mathrm{rad}/\mathrm{s}\right)(0.6 \ \mathrm{m}) = 2.4 \ \mathrm{m}/\mathrm{s}$$

If the particle were free to leave the circumference of the wheel, it would fly off in a straight-line tangent to the circumference at $2.4 \ \mathrm{m}/\mathrm{s}$.

(b) The linear speed at 0.3 m from the center of the wheel is

$$v = \omega r = \left(4.0 \ \mathrm{rad}/\mathrm{s}\right)(0.3 \ \mathrm{m}) = 1.2 \ \mathrm{m}/\mathrm{s}$$

Notice that radians have no dimensions and are therefore dropped out in arriving at the linear speed.

CENTRIPETAL ACCELERATION

(*College Physics* 9th ed. pages 207–208/10th ed. pages 211–213)

When a body is moving in a circular path with constant speed, its velocity is constantly changing because the direction of the tangential velocity is constantly changing. It is for this reason the body experiences a constant acceleration. The acceleration produces a change in direction but not in speed. Therefore the acceleration must always be at right angles to the motion, since any component in the direction of the motion would produce a change in speed. The acceleration is always directed toward the center of the circle. Such an acceleration is called a *centripetal acceleration*. Centripetal means *center-seeking*.

AP Tip

Any body that travels along a curved path always experiences a centripetal acceleration.

Centripetal acceleration, a_c, is determined from the radius of curvature of the path and the speed as

$$\text{centripetal acceleration} = \frac{(\text{tangential speed})^2}{\text{radius of circular path}}$$

$$a_c = \frac{v^2}{r}$$

The SI units of centripetal acceleration are m/s^2, the same as any other acceleration.

CENTRIPETAL FORCE

(*College Physics* 9th ed. pages 209–214/10th ed. pages 214–219)

According to Newton's second law of motion, any object experiencing acceleration by an unbalanced force, then the force is proportional to the acceleration and the direction of the acceleration. The unbalanced force that causes centripetal acceleration is called *centripetal force* and is directed toward the center of curvature.

AP Tip

A body moving with uniform speed in a circle is not in equilibrium.

Centripetal force is

$$F_c = ma_c = \frac{mv^2}{r} = m\omega^2 r$$

The SI unit of centripetal force, as with all forces, is the newton, N.

Any body moving in a curved path *always* experiences centripetal force. Some agent supplies the centripetal force. A ball tied to a string that is twirled overhead in a flat circle experiences centripetal force; so does the diesel engine rounding a curve on railroad tracks; so does the Earth in its orbit about the Sun. The tension in the string is the agent that supplies the centripetal force on the ball. The rails provide centripetal force to the diesel engine. The force of gravity is the agent providing the centripetal force on the Earth.

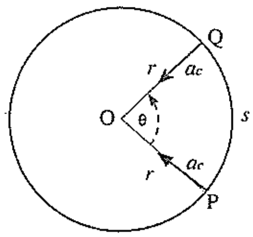


Diagram 1

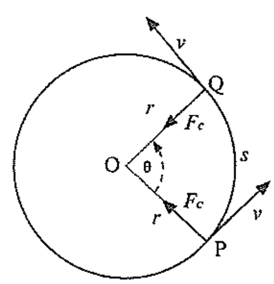


Diagram 2

In Diagram 1, a particle, as seen from above, travels with uniform speed v counterclockwise in a horizontal circle of radius r. As it moves from point P to point Q it undergoes an angular displacement θ and subtends an arc length s. Regardless of its location, the particle experiences a centripetal acceleration a_c directed along the radius toward the center of the circle, O.

In Diagram 2, the particle experiences a centripetal force F_c directed along the radius to the center of the circle. At point P the particle experiences a tangential velocity v. A tangent of course touches a curve at one point and one point only. Note the direction of the tangential velocity at point Q. The speed remains the same but the direction changes. The tangential velocity vector is always perpendicular to the centripetal acceleration vector and the centripetal force vector.

If the centripetal force were suddenly removed at point Q, no outside force acts on the particle and it obeys Newton's first law of motion and flies away in a straight line with speed v at some direction.

SAMPLE PROBLEM 4

A 2000-kg car rounds a flat curve of radius 60 m with a velocity of 9 m/s.

(a) What centripetal force acts on the car?

(b) What agent provides the centripetal force?

SOLUTION TO PROBLEM 4

(a) Centripetal force:

$$F_c = \frac{mv^2}{r} = \frac{(2000 \text{ kg})\left(9 \text{ m/s}\right)^2}{60 \text{ m}} = \mathbf{2.7 \times 10^3 \text{ N}}$$

(b) The agent providing the force is the friction between road surface and the tires.

SAMPLE PROBLEM 5

A 500-g wooden ball is attached to a piece of string rated with a maximum tension of 8 N. The ball is tied to a 1 m piece of the string and is whirled overhear in a horizontal circle. What maximum speed can the ball have?

SOLUTION TO PROBLEM 5

The centripetal force cannot exceed 8 N. Since $F_c = \frac{mv^2}{r}$, then

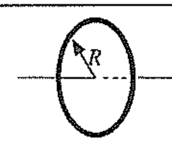
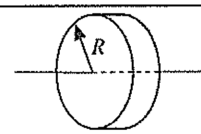
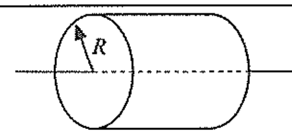
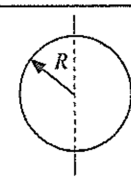
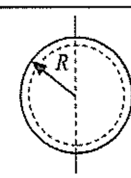
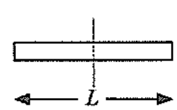
$v^2 = \frac{F_c r}{m}$ and taking the square root of both sides gives

$$v = \sqrt{\frac{F_c r}{m}} = \sqrt{\frac{(8 \text{ N})(1 \text{ m})}{0.5 \text{ kg}}} = \mathbf{4 \text{ m/s}}$$

MOMENT OF INERTIA

(*College Physics* 9th ed. page 251/10th ed. page 257)

All rotating bodies have a property called a *moment of inertia*. The moment of inertia, I, of a body is a measure of the rotational inertia of the body. It depends upon the size, shape and mass distribution of the body with respect to an axis of rotation. Moment of inertia is a scalar quantity with units of $\text{kg} \cdot \text{m}^2$. The moments of inertia of several bodies are given below. In most cases, the equations are derived by using the calculus.

Object	**Moment of Inertia**	**Shape**
Hoop of mass M and radius R. The axis of rotation is through the geometric center.	$I = MR^2$	
Solid disk of mass M and radius R. The axis of rotation is through the geometric center.	$I = \frac{1}{2}MR^2$	
Solid cylinder of mass M and radius R. The axis of rotation is through the geometric center.	$I = \frac{1}{2}MR^2$	
Solid sphere of mass M and radius R. The axis of rotation is through the center of mass.	$I = \frac{2}{5}MR^2$	
Hollow sphere of mass M and radius R. The axis of rotation is through the center of mass.	$I = \frac{2}{3}MR^2$	
Thin rod of mass M and length L. The axis of rotation is through the center of mass and is perpendicular to the length.	$I = \frac{1}{12}ML^2$	

SAMPLE PROBLEM 6

Determine the moment of inertia about an axis through the center of a 25 kg solid sphere whose diameter is 0.30 m.

SOLUTION TO PROBLEM 6

Since the diameter is given as 0.30 m, the radius is 0.15 m. Write the equation for the moment of inertia of a solid sphere and make substitution

$$I = \frac{2}{5}MR^2 = 0.4(25\text{ kg})(0.15\text{ m})^2 = \mathbf{0.23\ kg \cdot m^2}$$

THE PARALLEL-AXIS THEOREM

(*College Physics* 9th ed. pages 249–250/10th ed. pages 255–256)

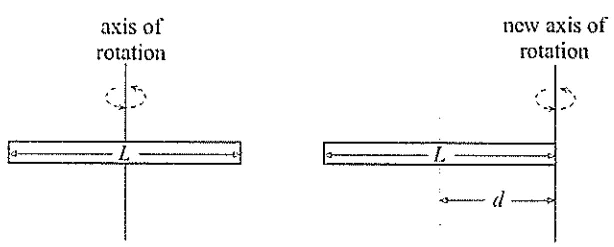
There are times when it is desirable to find the moment of inertia of a body about an axis other than its common geometric axis. To do so we make use of the *parallel axis theorem*. Let the moment of inertia of an object through its center of mass be I_c. The moment of inertia through any other axis parallel to the first is given by

$$I = I_c + Md^2$$

where M is the total mass of the object and d is the distance between the two parallel axes. Essentially, we are adding Md^2 to the moment of inertia through the center of mass of the body.

SAMPLE PROBLEM 7

A thin, uniform rod of length L and mass M rotates about an axis perpendicular to and through the center of the rod. Find the moment of inertia if the rod rotates about an axis at the end of the rod.



SOLUTION TO PROBLEM 7

The moment of inertia through the center of mass of a long, thin rod is $\frac{1}{12}MR^2$. The radius of rotation is $R = L$ and the moment of inertia about an axis at the end of the rod is $I = I_c + Md^2$. The distance from the original axis of rotation to the new one is $d = \frac{L}{2}$. Making substitution

$$I = I_c + Md^2 = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{12}ML^2 + \frac{1}{4}ML^2 = \frac{4}{12}ML^2$$

$$I = \frac{1}{3}ML^2$$

TORQUE AS THE AGENT OF ROTATION

(*College Physics* 9th ed. pages 247–248/10th ed. pages 252–254)

The expression *moment of force* and the word *torque* are synonymous, and are used interchangeably. We know that a force is necessary to change the motion of a body, that is, to produce an acceleration. If the rotation of a body about an axis is to be changed, torque τ about that axis must be applied. Recall that a torque tends to cause rotation. Unbalanced torque causes angular acceleration. The angular acceleration produced by a given torque depends upon the mass as well as the *distribution of mass* with respect to the axis, the moment of inertia.

Torque = moment of inertia x angular acceleration

A resultant torque, $\vec{\tau}$, acting on a rigid body of moment of inertia, I, about an axis, produces an angular acceleration, $\vec{\alpha}$.

$$\vec{\tau} = I\vec{\alpha}$$

Torque is a vector quantity. Bodies rotating counterclockwise have a positive (+) torque and clockwise rotations are negative (–).

Earlier we defined torque as the product of the force $\vec{F}$ applied at a distance $\vec{R}$ from the axis of rotation. Recall that we called $\vec{R}$ the moment arm. When $\vec{F}$ and $\vec{R}$ are perpendicular

$$\vec{\tau} = \vec{R}\vec{F} \text{ and } I\alpha = RF$$

When $\vec{F}$ and $\vec{R}$ are not perpendicular,

$$\vec{\tau} = \vec{R}\vec{F}\sin\theta$$

SAMPLE PROBLEM 8

A pulley attached to an electric motor is a uniform disk of mass 1.2 kg and radius 0.12 m. The pulley is spinning at 1200 rpm when the motor is turned off. The pulley uniformly slows to rest in 40 s. What torque brings the pulley to rest?

SOLUTION TO PROBLEM 8

First we need to determine the angular acceleration.

$$\omega_0 = 1200 \ ^{\text{rev}}\!/_{\text{min}} \times {}^{1\text{ min}}\!/_{60\text{ s}} \times {}^{2\pi \text{ rad}}\!/_{1\text{ rev}} = 126 \ ^{\text{rad}}\!/_{\text{s}}$$

$$\text{Angular acceleration } \alpha = \frac{\omega - \omega_0}{t} = \frac{0 - 126 \ ^{\text{rad}}\!/_{\text{s}}}{40\text{ s}} = -3.2 \ ^{\text{rad}}\!/_{\text{s}^2}$$

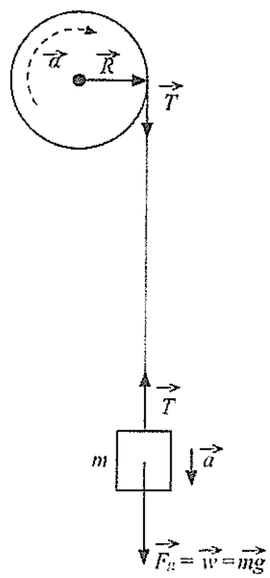
The required torque is $\tau = I\alpha$ and $I = \frac{1}{2}MR^2$. Combining equations

$$\tau = \frac{1}{2}\alpha MR^2 = (0.5)\left(-3.2 \ \frac{\text{rad}}{\text{s}^2}\right)(1.2\text{ kg})(0.12\text{ m})^2 = \mathbf{-0.028\ N \cdot m}$$

The negative sign appears because the force must be directed opposite to the direction of rotation of the pulley.

Sample Problem 9

A 0.6 kg mass, as shown in the diagram below, hangs at rest from the end of a cord wrapped several times around a pulley of 0.15 m radius. When released from rest, the mass falls 2.2 m in 6.0 s. Determine the moment of inertia of the pulley. Ignore the mass of the cord, friction, and air resistance.



Solution to Problem 9

The torque acting on the pulley is $\vec{\tau} = I\vec{\alpha}$ and the force acting on the mass is $\vec{F} = m\vec{a}$. Next we find the acceleration a of the system. We know the vertical drop from rest and the time it takes for the drop, $y = -2.2\text{ m}$ and $t = 6\text{ s}$. Recall that

$$y = v_0 t + \frac{1}{2}at^2 = 0 + \frac{1}{2}at^2$$

Solving for a, $a = \frac{2y}{t^2} = \frac{2(-2.2\text{ m})}{(6\text{ s})^2} = -0.12\ \text{m}/\text{s}^2$

The negative sign implies that m is accelerating downward.

The angular acceleration, α, of the pulley is related to a by $a = \alpha R$ and then

$$\alpha = \frac{a}{R} = \frac{\left(-0.12\ \text{m}/\text{s}^2\right)}{0.15\text{ m}} = -0.8\ \text{rad}/\text{s}^2$$

The negative sign means the pulley accelerates in the clockwise direction.

From Newton's second law, the unbalanced force acting on mass m is $\vec{F}_{net} m\vec{a}$ and $T - mg = ma$. Solving for T,

$$T = ma + mg = m(a+g) = (0.6\text{ kg})\left(-0.12\ \text{m}/\text{s}^2 + 9.8\ \text{m}/\text{s}^2\right) = 5.8\text{ N}$$

Now we write $\tau = I\alpha$ for the pulley.

$$\tau = RF = RT = Ia$$

Solving for I,

$$I = \frac{RT}{\alpha} = \frac{(0.15\text{ m})(-5.8\text{ N})}{\left(-0.8\text{ rad}/\text{s}^2\right)} = \mathbf{1.1\ kg \cdot m^2}$$

Rotational Energy

(*College Physics* 9th ed. pages 254–256/10th ed. pages 259–262)

All moving bodies have kinetic energy. Rotating bodies have what we call *rotational kinetic energy*, K_{rot}, and we define it as $K_{rot} = \frac{1}{2}I\omega^2$. Rotational kinetic energy is a scalar and has the J as its unit.

Sample Problem 10

A 12 kg solid steel disk has a radius of 0.06 meter. The disk spins with a velocity of 10 rad/s. What is the rotational kinetic energy of the disk?

Solution to Problem 10

First, write the moment of inertia of the disk. A solid disk has $I = \frac{1}{2}MR^2$. Substitute this expression into $K_{rot} = \frac{1}{2}I\omega^2$.

$$\begin{aligned} K_{rot} &= \frac{1}{2}I\omega^2 \\ &= \frac{1}{2}\left(\frac{1}{2}MR^2\right)\omega^2 \\ &= \frac{1}{4}MR^2\omega^2 \\ &= (0.25)(12\text{ kg})(0.06\text{ m})^2\left(10\ \frac{\text{rad}}{\text{s}}\right)^2 \end{aligned}$$

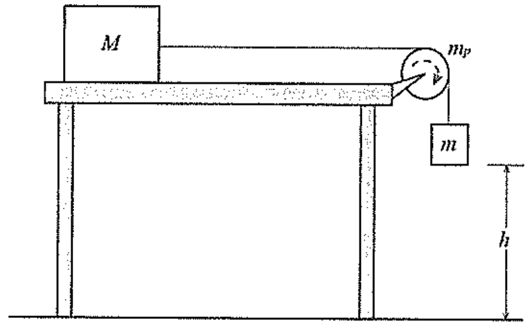
$$K_{rot} = \mathbf{1.08\ J}$$

Notice how we combined equations in this problem.

Consider the following problem. To this point we have ignored the mass and radius of pulleys. We have reached the point where we can treat them more realistically.

Sample Problem 11

A mass M is free to slide without friction across a horizontal tabletop. This mass is connected by a light string to a mass m that hangs over the edge of the table. The connecting string passes over a frictionless pulley in the shape of a disk with radius R and mass m_p. Find the velocity of the falling mass as it strikes the floor. The mass m, starting from rest, falls a distance h to the floor.



SOLUTION TO PROBLEM 11

The potential energy of the falling mass is converted into kinetic energy of all three masses. The kinetic energy of the pulley is due to its rotation. Because the pulley is a disk, its moment of inertia is $I = \frac{1}{2} m_p R^2$. Since frictionless forces are ignored, we use the conservation of mechanical energy. The gravitational potential energy of mass m is going to change into the kinetic energy due to translation of both masses plus the kinetic energy due to the rotation of the pulley. From the conservation of mechanical theorem we write

$$U_0 + K_0 = U + K$$

$$mgh + 0 = 0 + \frac{1}{2}mv^2 + \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

Here, v is the translational velocity of the blocks and ω is the rotational velocity of the pulley. At any instant M does not change height, only m appears in the gravitational potential energy term. Provided the string does not slip over the pulley, the tangential velocity of the edge of the pulley is also v. Thus, we can write $v = \omega R$ and substitute $\omega = v/R$. Making substitutions,

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}Mv^2 + \frac{1}{2}\left(\frac{1}{2}m_p R^2\right)\left(\frac{v}{R}\right)^2$$

Solving for v yields

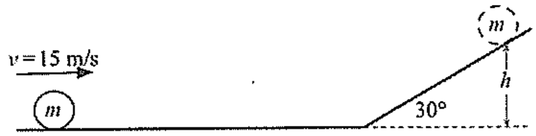
$$v = \sqrt{\frac{2mgh}{m + M + {}^{m_p}\!/_2}}$$

Bodies can have *rolling motion*. They can be traveling along the horizontal and be rolling at the same time. Such bodies will have both rotational motion as well as linear motion. The total kinetic energy of such a body is

$$K_{total} = K_{rot} + K = \frac{1}{2}I\omega^2 + \frac{1}{2}Mv^2$$

SAMPLE PROBLEM 12

A solid sphere of mass 10 kg rolls without slipping across a horizontal surface at 15 m/s and rolls up an inclined plane tilted at 30°. If friction losses are negligible, at what height, *h*, above the floor will the ball come to rest?



SOLUTION TO PROBLEM 12

The rotational and translational kinetic energies of the rolling sphere at the bottom of the incline will be totally changed into gravitational potential energy when it stops. Again we employ the law of conservation of mechanical energy. With friction being ignored, we write $U_0 + K_0 = U + K$ and

$$0 + K_{rot} + K = mgh + 0 + 0$$

$$\frac{1}{2}I\omega^2 + \frac{1}{2}mv^2 = mgh$$

The moment of inertia of the solid sphere is $I = \frac{2}{5}mR^2$ and the linear velocity of the sphere at its periphery is $v = \omega R$. Solving for angular velocity we get $\omega = v/R$. Substituting

$$\frac{1}{2}\left(\frac{2}{5}mR^2\right)\left(\frac{v}{R}\right)^2 + \frac{1}{2}mv^2 = mgh$$

Note that the mass of the sphere is common to both sides and divides out. The R^2 terms likewise divide out. Solving for *h*,

$$\frac{1}{5}v^2 + \frac{1}{2}v^2 = gh$$

And,

$$h = \frac{0.7v^2}{g} = \frac{0.7\left(15\ \mathrm{m/s}\right)^2}{9.8\ \mathrm{m/s^2}} = \mathbf{16\ m}$$

Notice that the mass *m*, radius *R* and the angular velocity ω did not enter into the calculations.

ROTATIONAL WORK AND POWER

(*College Physics* 9th ed. pages 254–256/10th ed. pages 260–262)

At an earlier time we defined work as the product of a displacement and the component of the force in the direction of the displacement. The rotational analogy is that *rotational work* is defined as the product of an angular displacement and the torque causing the rotation.

$$W = \tau\theta$$

Mechanical energy developed in machines is usually transmitted in the form of rotational work. Engine output is expressed in terms of the rate at which rotational work is done. The rate at which work is done is called power, P.

$$P = \frac{W}{t} = \frac{\tau\theta}{t}$$

Since average angular speed, $\bar{\omega}$, is θ/t, then power output is also expressed as

$$P = \tau\bar{\omega}$$

Work and power are both scalars. The unit of rotational work is the J and power is the W.

Sample Problem 13

A wheel, initially at rest, has a moment of inertia of $3.2\ \text{kg}\cdot\text{m}^2$ and is attached to a 1.0 kW electric motor. (a) What is the angular speed developed in the wheel 10 s after the motor is turned on? (b) What torque is developed by the motor?

Solution to Problem 13

(a) The work done by the motor in 10 s $= K_{rot}$ of the wheel after 10 s

$$W = Pt = \frac{1}{2}I\omega^2$$

Solving for ω

$$\omega = \sqrt{\frac{2Pt}{I}}\sqrt{\frac{2\left(1000\ {}^{\text{J}}\!/\!_{\text{s}}\right)(10\ \text{s})}{3.2\ \text{kg}\cdot\text{m}^2}} = \mathbf{79\ rad/s}$$

(b) Torque is $\tau = I\alpha = I\left(\frac{\omega - \omega_0}{t}\right) = \left(3.2\ \text{kg}\cdot\text{m}^2\right)\left(\frac{79\ \text{rad/s} - 0}{10\ \text{s}}\right) = \mathbf{25\ N\cdot m}$

Angular Momentum

(*College Physics* 9th ed. pages 257–261/10th ed. pages 262–267)

Any rigid body that rotates has *angular momentum*, $\vec{L}$, and its angular momentum is $\vec{L} = I\vec{\omega}$. Bodies that travel along a curved path also have angular momentum about a given point $\vec{L} = m\vec{v}\vec{r}$.

The unit of angular momentum is the $kg\cdot m^2/s$.

AP Tip

The angular momentum is a vector quantity. Bodies rotating or orbiting counterclockwise have a positive (+) angular momentum and when rotations are negative (–), the rotation is clockwise.

Angular acceleration is defined as the time rate of change angular velocity. Torque then, may be defined as the product of the moment of inertia and the time rate of angular velocity or

$$\vec{\tau} = I\vec{\alpha} = I\frac{\Delta\vec{\omega}}{\Delta t}$$

Multiplying both sides of the equation by Δt gives $\vec{\tau}\Delta t = I\Delta\vec{\omega}$. The quantity $\vec{\tau}\Delta t$ is called angular impulse and $I\Delta\vec{\omega}$ is the change in the angular momentum, $\Delta\vec{L}$. Just as linear impulse changes linear momentum, angular impulse changes angular momentum.

$$\vec{\tau}\Delta t = I\Delta\vec{\omega} = \Delta\vec{L}$$

Sample Problem 14

A hollow sphere having a mass of 4 kg and a radius of 4 cm is set into motion about an axis through the center. If the sphere has an angular speed of 20 rad/s, what is its angular momentum?

Solution to Problem 14

The moment of inertia of a hollow sphere is $\frac{2}{3}MR^2$. Its angular momentum is

$$\begin{aligned}L &= I\omega \\ &= \left(\frac{2}{3}MR^2\right)\omega^2 \\ &= \frac{2(4\text{ kg})(0.04\text{ m})^2\left(20\ \text{rad}/\text{s}\right)^2}{3} \\ &= \mathbf{1.7\ kg\cdot\frac{m^2}{s}}\end{aligned}$$

Sample Problem 15

A grinding wheel mounted on the shaft of an electric motor is initially at rest. The wheel has a mass of 8 kg and a radius of 0.12 m. Twenty seconds after the motor was turned on, the grinding wheel reached its maximum speed of 1800 rpm. What was the torque developed?

Solution to Problem 15

First, we convert 1800 rpm to rad/s.

$$\omega = 1800\text{ rpm}\times\frac{1\text{ min}}{60\text{ s}}\times\frac{2\pi\text{ rad}}{1\text{ rev}} = 188.5\ \text{rad}/\text{s}$$

Since the wheel starts from rest, $\omega_0 = 0$. Next, we write $\vec{\tau}\Delta t = I\Delta\vec{\omega}$.

The grinding wheel is a solid disk and $I = \frac{1}{2}MR^2$. Substituting and solving for the torque,

$$\tau = \frac{I\Delta\omega}{\Delta t}$$

$$= \frac{\frac{1}{2}MR^2(\omega-\omega_0)}{t} = \frac{(0.5)(8\text{ kg})(0.12\text{ m})^2\left(188.5\ \text{rad}/\text{s} - 0\right)}{(20\text{ s})}$$

$$= \mathbf{0.5\ N\cdot m}$$

THE LAW OF CONSERVATION OF ANGULAR MOMENTUM

(*College Physics* 9th ed. pages 257–261/10th ed. pages 262–267)

If no external torque acts on a body, the angular momentum of a body rotating about a fixed axis is constant.

In a system where there is no external torque, the total angular momentum before any event is always equal to the total angular momentum after the event. We call this statement the *law of conservation of angular momentum.*

$$\Sigma\vec{L} = \Sigma\vec{L}_0$$

$$I_0\omega_0 = I\omega$$

SAMPLE PROBLEM 16

A neutron star is formed when a star, such as our Sun, collapses in on itself. Before collapse, the mass of the star is M_0 and its radius R_0. After collapse, the neutron star has mass M_0 and radius $(1\times10^{-5})R_0$. The mass does not change but the radius shrinks by a factor of one hundred thousand. Before collapse, the star rotated at 1 revolution every 25 days. Determine the rotation rate of the neutron star in rev/s.

SOLUTION TO PROBLEM 16

Angular momentum must be conserved and the total angular momentum before collapse must equal the total angular momentum after collapse. We write the law of conservation of angular momentum $I_0\omega_0 = I\omega$.

$$\frac{1}{2}M_0R_0^2\omega_0 = \frac{1}{2}M_0\left(1\times10^{-5}R_0\right)^2\omega$$

Note that the one-half, M_0 and R_0^2 appear on both side of the equation and divide out.

$$\omega = \frac{\omega_0}{\left(1\times10^{-5}\right)^2} = \frac{\frac{1\text{ rev}}{25\text{ d}}\times\frac{1\text{ d}}{24\text{ h}}\times\frac{1\text{ h}}{3600\text{ s}}}{\left(1\times10^{-10}\right)} = \mathbf{4630\ \frac{rev}{s}}$$

It is important to keep in mind that it is $I\omega$, the product of the moment of inertia and the angular velocity that is conserved, and not the angular velocity ω. In many situations, internal rearrangement of the masses of a system may change its moment of inertia. When that happens the angular velocity changes even though no external torque is applied to the system.

A figure skater starts a pirouette in a crouch, rotating on the toe of one skate with the other leg and both arms extended. She then slowly rises, pulling the extended leg and arms to her body, thus reducing her moment of inertia about the axis of rotation. As she does so, her angular velocity increases substantially.

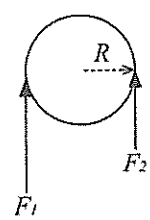
The torque due to friction between the ice skate and ice is quite small. As a result, the angular momentum remains nearly constant.

Circular Motion and Rotational Motion: Student Objectives for the AP Exam

- You should be able to discuss why a particle moving in a horizontal circle at constant speed experiences a centripetal acceleration and a centripetal force.
- You should be able to explain the relationship between angular and linear descriptions of rotational motion.
- You should be able to explain how to locate experimentally the c.m. of a flat body of irregular shape.
- You should explain the meaning of the term moment of inertia.
- You should be able to explain why an ice skater spins faster when she pulls her arms in toward her body than when she extends her arms.

Multiple-Choice Questions

1. A cylinder is rotating clockwise about a frictionless axle when two forces are applied to the rim of the cylinder as shown below.



The cylinder will rotate with

(A) increasing angular speed in the clockwise direction since $|\vec{F}_1| > |\vec{F}_2|$

(B) decreasing angular speed in the clockwise direction since the net force acting on the cylinder is $|\vec{F}_{net}| = |\vec{F}_1| - |\vec{F}_2|$

(C) decreasing angular speed in the clockwise direction since $|\vec{\tau}_1| > |\vec{\tau}_2|$

(D) increasing angular speed in the clockwise direction since $|\vec{\tau}_1| > |\vec{\tau}_2|$