6

MOMENTUM, IMPULSE, AND COLLISIONS

LINEAR MOMENTUM

(*College Physics* 9th ed. pages 167–169/10th ed. pages 170–172)

Linear momentum, or simply momentum, $\vec{p}$, is a characteristic of a moving body. It is a vector quantity with magnitude and direction. It is defined as the product of the mass, *m*, of the body and its velocity, $\vec{v}$.

$$\vec{p} = m\vec{v}$$

The SI unit of momentum is $\text{kg}\cdot\frac{\text{m}}{\text{s}}$.

All moving bodies have two properties, (1) momentum and (2) kinetic energy. Stationary bodies do not have these properties.

SAMPLE PROBLEM 1

(a) Find the momentum of a proton, $m_p = 1.67\times10^{-27}$ kg, that has been accelerated to a velocity of 4.50×10^{6} m/s in a particle accelerator.

(b) What work, in eV, was done by the particle accelerator in giving its momentum?

SOLUTION TO PROBLEM 1

(a) Momentum is $p = mv$ and

$$p = (1.673\times10^{-27}\ \text{kg})(4.50\times10^{6}\ \text{m/s}) = \mathbf{7.53\times10^{-21}\ kg\cdot m/s}$$

(b) The work done by the accelerator is the kinetic energy of the proton

$$W = K = \frac{1}{2}mv^2$$

$$W = \frac{1}{2}(1.673\times10^{-27}\ \text{kg})(4.5\times10^{6}\ \text{m/s})^2$$

$$= 1.69\times10^{-14}\ J \times \frac{1\ eV}{1.60\times10^{-19}\ J}$$

$$= \mathbf{1.06\times10^{5}\ eV}$$

SAMPLE PROBLEM 2

Derive an expression relating momentum, p, and kinetic energy, K, as $p = f(K)$.

SOLUTION TO PROBLEM 2

$K = \frac{1}{2}mv^2$ and $K = \frac{1}{2}(mv)v$ then $K = \frac{1}{2}pv$. Solving for p yields $p = \frac{2K}{v}$. Use the data from Sample Problem 1 and determine the momentum of the proton from its kinetic energy.

$$p = \frac{2K}{v} = \frac{2(1.69\times10^{-14}\ \text{J})}{(4.50\times10^{6}\ \text{m/s})} = \mathbf{7.53\times10^{-21}\ kg\cdot m/s}$$

IMPULSE

(*College Physics* 9th ed. pages 167–169/10th ed. pages 170–172)

Consider when a golf ball is struck with a golf club. The club is in contact with the golf ball for a very short time interval, Δt. During this time interval, a very large force is exerted on the ball; this force varies with time in a very complex manner that is difficult to determine. Forces of this kind are called *impulsive forces*. Initially, the golf ball is at rest. Over the time interval Δt the ball is accelerated and separates from the club with some velocity, v.

We define *impulse*, J, as a vector quantity equal to the product of the force, $\vec{F}$, and the time interval, Δt over which it acts. The direction of the impulse is the same as that of the force.

$$\vec{J} = \vec{F}\Delta t$$

The unit of impulse is the $\text{N}\cdot\text{s}$.

Impulse causes change in momentum, and we can write $\vec{J} = \Delta\vec{p}$, or

$$\vec{F}\Delta t = m\Delta\vec{v}$$

With every momentum change there is an impulse.

Sample Problem 3

A 1100 kg car traveling at 10 m/s collides with a concrete barrier and comes to rest in 0.9 second.

(a) What force does the barrier exert on the car?

(b) What acceleration does the car experience?

Solution to Problem 3

(a) The car undergoes a momentum change that is caused by impulse, $m\Delta v = F\Delta t$. Since we are seeking the force we solve for F

$$F = \frac{m\Delta v}{\Delta t} = \frac{m(v - v_0)}{\Delta t} = \frac{(1100\text{ kg})(0 - 10\ \text{m/s})}{(0.9\text{ s})} = \mathbf{-12.2 \times 10^3\ N}$$

(b) Unbalanced forces cause accelerations: $F = ma$. The acceleration is

$$a = \frac{F}{m} = \frac{-12.2 \times 10^3\text{ N}}{1100\text{ kg}} = \mathbf{-11.1\ m/s^2}$$

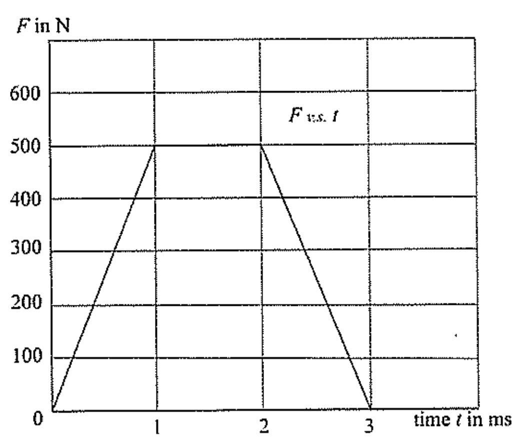
We could have also found the acceleration from

$$a = \frac{v - v_0}{t} = \frac{0 - 10\ \text{m/s}}{0.9\text{ s}} = \mathbf{-11.1\ m/s^2}$$

> **AP Tip**
>
> The area beneath an F *vs.* Δt graph for a collision is the impulse generated during the collision.

Sample Problem 4

A cue ball traveling the +*x*-axis makes a *head-on* collision with a pool ball. The force developed over time period *t* in the collision is shown in the graph below. From the graph, determine the impulse generated in the collision.



SOLUTION TO PROBLEM 4

The impulse $J = F\Delta t$ is the area under the curve that consists of two triangles and a rectangle.

$$J = A = \frac{1}{2}(base)(altitude) + (base)(altitude) + \frac{1}{2}(base)(altitude)$$

$$J = \frac{1}{2}(1.00\times10^{-3}\ \text{s})(500\ \text{N}) + (1.00\times10^{-3}\ \text{s})(500\ \text{N}) + \frac{1}{2}(1.00\times10^{-3}\ \text{s})(500\ \text{N})$$
$$= \mathbf{1.00\ N\cdot s}$$

The direction is to the right.

LAW OF CONSERVATION OF LINEAR MOMENTUM

(*College Physics* 9th ed. pages 172–175/10th ed. pages 176–179)

In a closed system with no external forces acting, the total initial momentum, $\Sigma\vec{p}_0$ of the system is always equal to the total final momentum, $\Sigma\vec{p}$ of the system. This is a statement of one of the most important principals in all of physics, the *Law of Conservation of Linear Momentum*. Or

$$\Sigma\vec{p}_0 = \Sigma\vec{p}$$

SAMPLE PROBLEM 5

A 20 gram projectile is shot horizontally into a stationary 7 kg block of wood and becomes imbedded in it. Immediately after impact the block is observed to move to the right with a velocity of 0.52 $\mathrm{m/s}$. Determine the initial velocity of the projectile.

SOLUTION TO PROBLEM 5

Anytime we have a collision we start by writing the law of conservation of linear momentum: $\Sigma p_0 = \Sigma p$. Before collision one body is moving, the projectile. After collision a single body moves, the block with the imbedded bullet. The total initial momentum equals the total final $mv_0 = (M+m)V$. Solving for V,

$$v_0 = \frac{(M+m)V}{m} = \frac{(7.020\ \text{kg})(0.52\ \mathrm{m/s})}{(0.020\ \text{kg})} = \mathbf{182.5\ m/s}$$

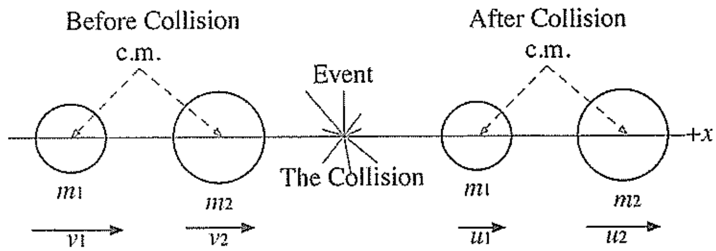
Note that the sign of the velocity is positive. The block + bullet combination moves to the right.

COLLISIONS

(*College Physics* 9th ed. pages 175–176/10th ed. pages 179–180)

When two bodies strike one another, a *collision* occurs, in which case the net external force is zero. During a collision an impulse is generated that produces momentum change. There will always be a momentum change and the law of conservation of linear momentum always holds true.

Consider two non-rotating spherical bodies traveling to the right in such a way that the centers of mass, c.m., travel along the x-axis as diagramed below. Let v's represent initial velocities and u's represent final velocities.



The bodies collide and then separate after collision with their centers of mass, c.m., continuing along the x-axis. Such a collision is called a one-dimensional collision or a head-on collision. Initially, two bodies are in linear motion and after the event, the collision, two bodies are in linear motion.

Any collision that is not a perfectly elastic collision is called inelastic. In all inelastic collisions there is a disappearance of kinetic energy during the collision process. A major portion of the lost kinetic energy is transformed into thermal energy—heat. The best way to account for the thermal energy loss in collisions is by means of an empirical quantity that we call the *coefficient of restitution*, ε. We define the coefficient of restitution as the ratio of the relative rate of separation after collision to the relative rate of approach before collision

$$\varepsilon = \frac{\text{relative rate of separation}}{\text{relative rate of approach}}$$

or

$$\varepsilon = \frac{u_2 - u_1}{v_1 - v_2}$$

Of course v's represent initial velocities of the colliding bodies and u's represent all final velocities.

The coefficient of restitution will have values ranging from $0 \le \varepsilon \le 1$. The lower the value of ε the more thermal energy that is generated in the collision. In the case where $\varepsilon = 0$, the collision is a totally inelastic collision and the bodies merge into a single composite body. This type of collision is called a *totally inelastic collision*. In a perfectly elastic collision, the relative rate of separation equals the relative rate of approach and $\varepsilon = 1$. When $0 < \varepsilon < 1$, the collision is a *partially elastic collision* where the bodies collide and separate and their shapes are somewhat altered in the collision process.

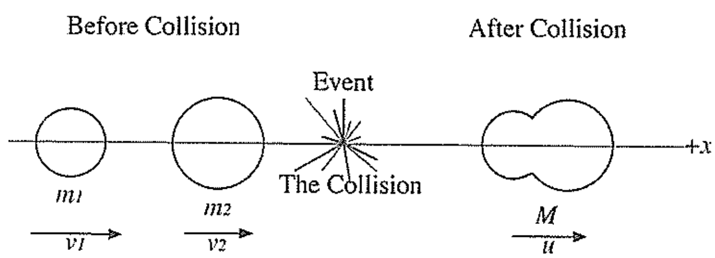
THE TOTALLY INELASTIC COLLISION

(*College Physics* 9th ed. pages 176–179/10th ed. pages 180–183)

In the totally inelastic collision the coefficient of restitution is zero, $\varepsilon = 0$, the initial kinetic energy is always greater than the final kinetic energy, $\Sigma K_0 > \Sigma K$, and the total initial momentum is always equal to the final momentum, $\Sigma p_0 = \Sigma p$. The law of conservation of linear momentum is always our starting point.

SAMPLE PROBLEM 6

A 4.0 kg mass, m_1 travels to the right along the x-axis with a velocity of $v_1 = 10.0 \text{ m/s}$. It overtakes a second mass $m_2 = 6.0$ kg whose center of mass travels to the right along the x-axis with a velocity of $v_2 = 4.0 \text{ m/s}$. The masses undergo a totally inelastic collision forming a composite mass, M. Calculate the velocity of the composite mass immediately after collision.



Totally Inelastic Collision

SOLUTION TO PROBLEM 6

As stated above, the starting point is the law of conservation of linear momentum: $\Sigma p_0 = \Sigma p$. Before the collision, two bodies are moving. In the collision process the bodies merge forming a single body. We write $m_1 v_1 + m_2 v_2 = Mu$. Next we solve for the final velocity, u, and substitute the known values.

$$u = \frac{m_1 v_1 + m_2 v_2}{M} = \frac{(4.0 \text{ kg})(10.0 \text{ m/s}) + (6.0 \text{ kg})(4.0 \text{ m/s})}{(10.0 \text{ kg})}$$

$$= \mathbf{6.4\ \frac{m}{s}}$$

In a totally inelastic collision, kinetic energy is not conserved and $\Sigma K_0 > \Sigma K$. Calculating both K_0 and K

$$\Sigma K_0 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

$$= \frac{1}{2}(4.0 \text{ kg})(10.0 \text{ m/s})^2 + \frac{1}{2}(6.0 \text{ kg})(4.0 \text{ m/s})^2 = \mathbf{248.0\ J}$$

$$\Sigma K = \frac{1}{2}Mu^2 = \frac{1}{2}(10.0\text{ kg})(6.4\text{ m/s})^2 = \mathbf{204.8\ J}$$

By the work-kinetic energy theorem

$$W = \Delta K = \Sigma K - \Sigma K_0 = 204.8\text{ J} - 248.0\text{ J} = \mathbf{-43.2\ J}$$

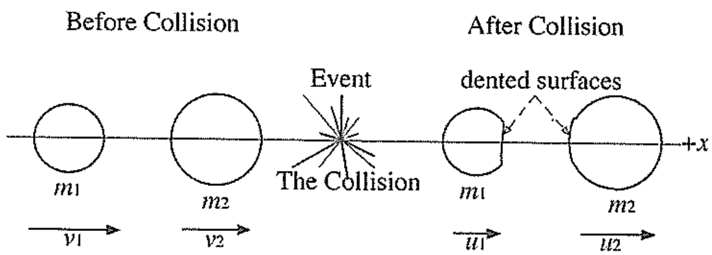
During the collision, work is done merging the bodies and thermal energy is developed in the merged mass, *M*. Sound, acoustical energy is produced.

THE PARTIALLY ELASTIC COLLISION

(*College Physics* 9th ed. pages 179–182/10th ed. pages 183–185)

SAMPLE PROBLEM 7

A 4.0 kg mass, m_1, travels to the right along the x-axis with a velocity of $v_1 = 10.0$ m/s. It overtakes a second mass $m_2 = 6.0$ kg whose center of mass travels to the right along the x-axis with a velocity of $v_2 = 4.0$ m/s. The masses undergo a partially perfectly elastic collision where the coefficient of restitution for the interacting bodies is $\varepsilon = 0.75$. Calculate the velocity of the masses m_1 and m_2 immediately after collision.



Partially Elastic Collision

SOLUTION TO PROBLEM 7

First we can write the law of conservation of linear momentum: $\Sigma p_0 = \Sigma p$ and $m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2$

Substituting without units

$$(4.0)(10.0) + (6.0)(4.0) = 4.0u_1 + 6.0u_2$$

Solving for u_1 and u_2 we write equation (1)

$$4.0u_1 + 6.0u_2 = 64.0 \qquad (1)$$

From the definition of the coefficient of restitution, $\varepsilon = \dfrac{u_2 - u_1}{v_1 - v_2}$,

and we can write

$$u_2 - u_1 = \varepsilon(v_1 - v_2)$$

Substituting without units

$$u_2 - u_1 = 0.75(10.0 - 4.0) = 4.5$$

Solving for u_1 and u_2 we write equation (2) as

$$u_1 - u_2 = -4.5 \qquad (2)$$

Equations (1) and (2) constitute a pair of simultaneous linear equations.

$$4.0u_1 + 6.0u_2 = 64.0 \qquad (1)$$

$$u_1 - u_2 = -4.5 \qquad (2)$$

Solving them algebraically we multiply equation (2) by 6

$$4.0u_1 + 6.0u_2 = 64.0$$

$$6.0u_1 - 6.0u_2 = -27.0$$

Adding

$$10.0u_1 = 37.0$$

Dividing both sides by 10 yields

$$u_1 = \mathbf{3.7\,\frac{m}{s}}$$

Substituting into equation (1)

$$(4.0)(3.7) + 6.0u_2 = 64.0$$

Solving for u_2

$$u_2 = \mathbf{8.2\,\frac{m}{s}}$$

The total initial total kinetic energy of the system is

$$\Sigma K_0 = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}(4.0\text{ kg})(10.0\text{ m/s})^2 + \frac{1}{2}(6.0\text{ kg})(4.0\text{ m/s})^2 = \mathbf{248.0\text{ J}}$$

The final total kinetic energy is

$$\Sigma K = \frac{1}{2}m_1 u_1^2 + \frac{1}{2}m_2 u_2^2$$

$$= \frac{1}{2}(4.0\text{ kg})(3.7\text{ m/s})^2 + \frac{1}{2}(6.0\text{ kg})(8.2\text{ m/s})^2 = \mathbf{229.1\text{ J}}$$

By the work-kinetic energy theorem

$$W = \Delta K = \Sigma K - \Sigma K_0 = 229.1\text{ J} - 248.0\text{ J} = \mathbf{-18.9\text{ J}}$$

THE PERFECTLY ELASTIC COLLISION

(*College Physics* 9th ed. pages 179–182/10th ed. pages 183–185)

In this perfectly elastic collisions, the quantity of energy before and after the event is unaltered. No energy is lost during this collision process. Such a collision is called a perfectly elastic collision.

In nature very few collisions are perfectly elastic. Neutrons and mono-atomic molecules such as helium approximate perfectly elastic collisions. Colliding ivory billiard balls come within several percent of being perfectly elastic. Other than the formality of deriving the equations of a perfectly elastic head-on collision, mathematical analysis of these collisions are actually approximations.

Let's look at a head-on perfectly elastic collision.

In perfectly elastic collisions, as in all collisions, the law of conservation of linear momentum, $\Sigma p_0 = \Sigma p$, always holds true and kinetic energy is also conserved, $\Sigma K_0 = \Sigma K$. It is only in the perfectly elastic collision that $\Sigma K_0 = \Sigma K$ holds true. For a perfectly elastic collision we define the coefficient of restitution, $\varepsilon = 1$.

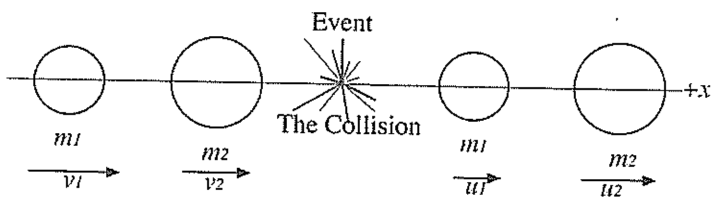
$$u_1 = \frac{(m_1 - m_2)}{(m_1 + m_2)} v_1 + \frac{2m_2}{(m_1 + m_2)} v_2$$

$$u_2 = v_1 - v_2 + u_1$$

$$u_2 = \frac{2m_1}{(m_1 + m_2)} v_1 + \frac{(m_2 - m_1)}{(m_1 + m_2)} v_2$$

Sample Problem 8

A 4.0 kg mass, m_1, travels to the right along the x-axis with a velocity of v_1 = 10.0 m/s. It overtakes a second mass m_2 = 6.0 kg whose center of mass travels to the right along the axis with a velocity of v_2 = 4.0 m/s. The masses undergo a perfectly elastic collision. Calculate the velocity of the masses m_1 and m_2 immediately after collision.



Perfectly Elastic Collision

Solution to Problem 8

In all collisions the law of conservation of linear momentum always holds true and in a perfectly elastic collision kinetic energy is also conserved. From both of these laws we derive the following equations that only hold for perfectly elastic collisions.

The velocity of mass m_1 immediately after collision is found by using

$$u_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_1 + \left(\frac{2m_2}{m_1 + m_2}\right) v_2$$

$$u_1 = \left(\frac{4.0\text{ kg} - 6.0\text{ kg}}{10\text{ kg}}\right)\left(10.0\ \frac{\text{m}}{\text{s}}\right) + \left(\frac{12\text{ kg}}{10\text{ kg}}\right)\left(4.0\ \frac{\text{m}}{\text{s}}\right) = \mathbf{2.8}\ \frac{\mathbf{m}}{\mathbf{s}}$$

$$u_2 = \left(\frac{2m_1}{m_1 + m_2}\right) v_1 + \left(\frac{m_2 - m_1}{m_1 + m_2}\right) v_2$$

$$u_2 = \left(\frac{8.0\text{ kg}}{10.0\text{ kg}}\right)\left(10.0\ \frac{\text{m}}{\text{s}}\right) + \left(\frac{2.0\text{ kg}}{10.0\text{ kg}}\right)\left(4.0\ \frac{\text{m}}{\text{s}}\right) = \mathbf{8.8}\ \frac{\mathbf{m}}{\mathbf{s}}$$

Calculating the total initial kinetic energy, ΣK_0

$$\Sigma K_0 = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$$

$$\Sigma K_0 = \frac{1}{2}(4.0\text{ kg})(10.0\text{ m/s})^2 + \frac{1}{2}(6.0\text{ kg})(4.0\text{ m/s})^2 = \mathbf{248.0\text{ J}}$$

Calculating the total finial kinetic energy, ΣK

$$\Sigma K = \frac{1}{2}m_1 u_1^2 + \frac{1}{2}m_2 u_2^2$$

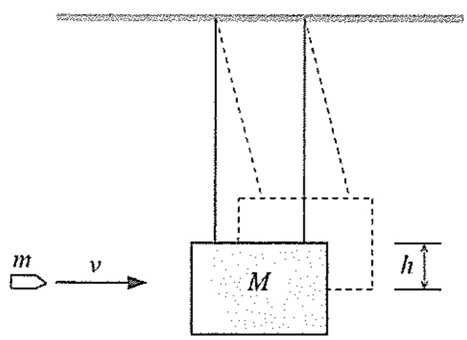
$$\Sigma K = \frac{1}{2}(4.0\text{ kg})(2.8\text{ m/s})^2 + \frac{1}{2}(6.0\text{ kg})(8.8\text{ m/s})^2 = \mathbf{248.0\text{ J}}$$

$\Sigma K_0 = \Sigma K$ and the collision is perfectly elastic.

BALLISTIC PENDULUM

(*College Physics* 9th ed. pages 178–179/10th ed. pages 182–183)

A ballistic pendulum is a device that is used to measure the velocities of small projectiles such as bullets. Ballistic pendulums in general consist of a block of wood suspended by vertical cords. When a bullet is fired into the wood block it transfers momentum and energy to the block causing it to swing through an arc. By measuring the mass of the block, M, the mass of the bullet, m, and by measuring the vertical elevation, h, of the block the initial velocity of the bullet can be determined.



SAMPLE PROBLEM 9

Consider the above diagram. A 10 g bullet is fired into the stationary 3990 g block of a ballistic pendulum. The bullet is captured in the block elevating it vertically by 0.03 m.

(a) What is the velocity, V, of the block-bullet system just after impact?

(b) What is the velocity of the bullet prior to impact with the block?

SOLUTION TO PROBLEM 9

(a) Mechanical energy is conserved from the time the bullet impacts the block through its 0.03 m rise. Writing the law of conservation of mechanical:

$E_0 = E$ and at the moment just after impact:

$\frac{1}{2}(m+M)V^2 = (m+M)gh$. Solving for V:

$$V = \sqrt{2gh} = \sqrt{2\left(9.8\ \mathrm{m/s^2}\right)(0.03\ \mathrm{m})} = \mathbf{0.77\ m/s}$$

(b) On impact, $\Sigma p_0 = \Sigma p$ and $mv = (m+M)V$. Solving for the initial velocity of the bullet, v:

$$v = \frac{(m+M)V}{m} = \frac{(4.00\ \mathrm{kg})\left(0.77\ \mathrm{m/s}\right)}{(0.01\ \mathrm{kg})} = \mathbf{308\ m/s}$$

CENTER OF MASS

(*College Physics* 9th ed. pages 151, 241–243/10th ed. pages 154, 246–248)

The *center of mass*, c.m., of a system is the point, either inside or outside the system, where the system can be balanced in a uniform gravitational field. All bodies and all system behave as if all of their masses were concentrated at the c.m.

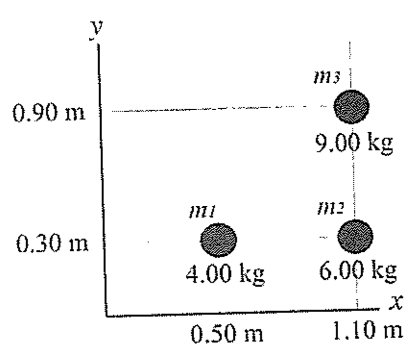
In two-dimensional space the center of mass of a system composed of i particles can be located from the coordinates of the c.m.

$$x_{\text{c.m.}} = \frac{\Sigma x_i m_i}{\Sigma m_i} \quad \text{and} \quad y_{\text{c.m.}} = \frac{\Sigma y_i m_i}{\Sigma m_i}$$

In a uniform gravitational field, the center of mass and the center of gravity coincide.

SAMPLE PROBLEM 10

Determine the center of mass for the following system.



SOLUTION TO PROBLEM 10

The x-coordinate of the c.m.

$$x_{\text{c.m.}} = \frac{\Sigma x_i m_i}{\Sigma m_i}$$

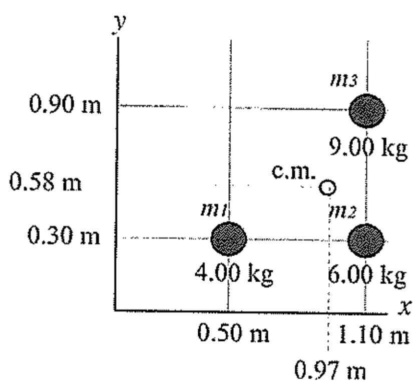
$$= \frac{(0.50\text{ m})(4.00\text{ kg}) + (1.100\text{ m})(6.00\text{ kg}) + (1.10\text{ m})(9.00\text{ kg})}{4.00\text{ kg} + 6.00\text{ kg} + 9.00\text{ kg}}$$

$$= \mathbf{0.97\text{ m}}$$

$$y_{\text{c.m.}} = \frac{\Sigma y_i m_i}{\Sigma m_i}$$

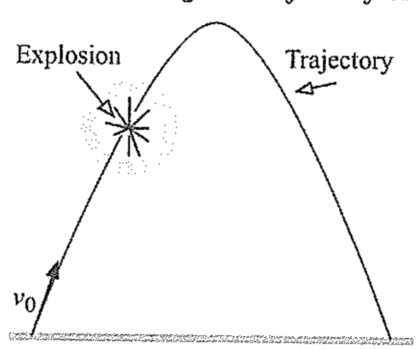
$$= \frac{(0.30\text{ m})(4.00\text{ kg}) + (0.30\text{ m})(6.00\text{ kg}) + (0.90\text{ m})(9.00\text{ kg})}{4.00\text{ kg} + 6.00\text{ kg} + 9.00\text{ kg}}$$

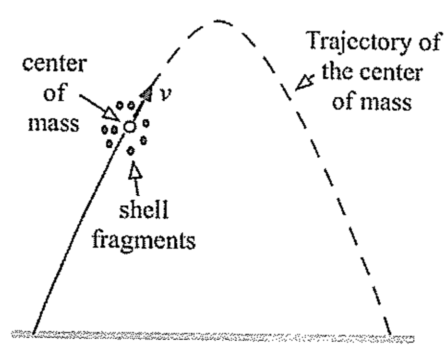
$$= \mathbf{0.58\text{ m}}$$



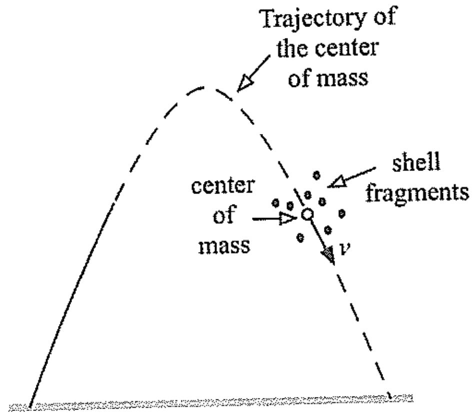
Consider an artillery shell fired from a cannon. Ignoring air resistance, the shell, actually the center of mass of the shell, follows a parabolic trajectory that is a function of the elevation angle and the initial velocity of the shell. At a point along the trajectory, as shown in the diagram below, the shell explodes into eight fragments.

The *cluster* of fragments form a cloud about the center of mass. The beauty of the law of conservation of linear momentum is that the c.m. follows the original trajectory of the shell.





The total momentum remains constant since no external forces act on the shell. The kinetic energy, however, increases because the shell fragments receive additional kinetic energy from the explosion.



Once one or more of the fragments impacts the ground, the center of mass of the system changes, and the trajectory shifts.

Momentum, Impulse, and Collision: Student Objectives for the AP Exam

- You should know how to determine the value of the average impulsive force if you know the change in the velocity of a body and the duration of the impulse.
- You should be able to explain how a system of two particles can have a total momentum of zero when both particles are moving.
- You should know the relationship between the coefficient of restitution and the total kinetic energy before and after a collision between two particles.
- You should know how to locate the c.m. for a system of particles.