3

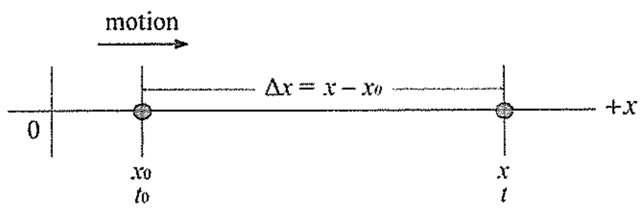
MOTION

VELOCITY

(*College Physics* 9th ed. pages 25–29/10th ed. pages 26–31)

The location of a particle traveling along the x-axis is described by its x-coordinate. The change in the position of the particle is its *displacement,* Δx. Initially, if the particle is located at position x_0 at time t_0 and at position x at time t, the particle is displaced by $\Delta x = x - x_0$. Recall that displacement is a vector. The elapsed time is $\Delta t = t - t_0$. For motion in one dimension we simply specify the displacement of the x-coordinate of the particle divided by the elapsed time. To the right of the origin, the coordinate is positive and to the left of the origin, the coordinate is negative. As an equation, we define the *average velocity,* $\bar{v}$, as the time rate of change of displacement, or

$$\bar{v} = \frac{x - x_0}{t - t_0} = \frac{\Delta x}{\Delta t}$$

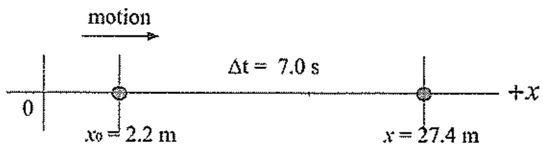


Average velocity is a vector and its SI unit is the $\mathrm{m/s}$.

SAMPLE PROBLEM 1

A particle travels along the positive x-axis. Initially, the particle is observed to be located at $x_0 = 2.2\ \text{m}$ to the right of the origin, and

7.0 s later the particle is at $x = 27.4$ m to the right of the origin. What is the average velocity of the particle?



Solution to Problem 1

The particle initially is located at the 2.2 m position and 7.0 s later it is positioned at the 27.4 m mark. We define average velocity with the equation

$$\bar{v} = \frac{x - x_0}{t - t_0} = \frac{27.4\text{ m} - 2.2\text{ m}}{7.0\text{ s}} = \mathbf{3.6\ m/s}$$

Between the two points the particle moves to the right at 3.6 $\mathrm{m/s}$.

If we start the particle at the origin, $x_0 = 0$, and if we start our stopwatches at $t_0 = 0$, then average velocity is

$$\bar{v} = \frac{x}{t}$$

Sample Problem 2

A proton has an average velocity of $4.5 \times 10^6\ \mathrm{m/s}$ at 0°. What is the displacement of the proton in time period of 2.0 μs ?

Solution to Problem 2

Since average velocity is expressed as $\bar{v} = x/t$, then displacement is $x = \bar{v}t$. Substituting we have

$$x = \bar{v}t = \left(4.5 \times 10^6\ \mathrm{m/s}\right)\left(2.0 \times 10^{-6}\ \mathrm{s}\right) = \mathbf{9.0\ m\ @\ 0^\circ}$$

The proton travels for 2.0 millionth of a second at 4.5 million meters per second. In that time period its displacement is 9.0 meters at 0°.

Sample Problem 3

A stock car is being driven on a circular track. The car starts at the start/finish line and makes one complete lap in 8 s. The track has a radius of 68 m. Determine the average velocity of the stock car.

Solution to Problem 3

Average velocity is defined as the displacement per unit time. The car leaves from the start/finish line and returns 8 s later completing a lap. Recall that displacement is the straight-line distance from the starting point to the finish point. The displacement is zero. Average velocity cannot be determined.

ACCELERATION

(College Physics 9th ed. pages 33–35/10th ed. pages 34–37)

Just as average velocity is the time rate of change of displacement, *average acceleration* is the time rate of change of velocity. As an equation

$$a = \frac{v - v_0}{t - t_0} = \frac{\Delta v}{\Delta t}$$

Constant acceleration mathematically behaves just like average acceleration. Acceleration is a vector that has units of $\mathrm{m/s^2}$ which is read as meters per second per second.

If we again start our observations at $t_0 = 0$, $a = \frac{v - v_0}{t}$ and $v - v_0 = at$, solving for the final velocity yields

$$v = v_0 + at$$

which states that when a particle moving at initial velocity v_0 undergoes an acceleration a for a time interval t, its final velocity is v.

Since the velocity of a particle is increasing or decreasing with time, we can express the average velocity for any time interval as the *arithmetic average* of the initial velocity v_0 and the final velocity v as

$$\bar{v} = \frac{v_0 + v}{2}$$

The average velocity is also $\bar{v} = \frac{x}{t}$. Equating these expressions for average velocity, $\bar{v} = \frac{x}{t} = \frac{v_0 + v}{2}$. Clearing of fractions gives $2x = (v_0 + v)t$. Since $v = v_0 + at$, we can replace final velocity v: $2x = (v_0 + v_0 + at)t = (2v_0 + at)t$. Dividing both sides by 2 yields the displacement of the body

$$x = v_0 t + \frac{1}{2}at^2$$

SAMPLE PROBLEM 4

A body traveling along the positive x-axis with an initial velocity of 10.0 m/s is uniformly accelerated to a velocity of 28.0 m/s over a time period of 6.0 seconds.

(a) What constant acceleration does the body experience?

(b) What is the displacement of the body over the 6.0 seconds of acceleration?

Solution to Problem 4

(a) Since we are seeking acceleration we will use the equation that defines it.

$$a = \frac{v - v_0}{t} = \frac{28.0 \ ^{m}\!/_{s} - 10.0 \ ^{m}\!/_{s}}{6.0 \text{ s}} = \mathbf{3.0} \ ^{\mathbf{m}}\!/_{\mathbf{s^2}}$$

Each and every second, the velocity of the body increases by 3.0 $^{m}\!/_{s}$.

The displacement of a body while undergoing constant accelerating is

(b) $x = v_0 t + \frac{1}{2} a t^2 = \left(10 \ ^{m}\!/_{s}\right)(6.0 \text{ s}) + (0.5)\left(3.0 \ ^{m}\!/_{s^2}\right)(6.0 \text{ s})^2 = \mathbf{114 \ m}$

Sample Problem 5

A particle travels along the $+x$-axis with a velocity of 12 $^{m}\!/_{s}$. It is accelerated at 2.5 $^{m}\!/_{s^2}$ for a time period of 15 s. What is the final velocity of the particle?

Solution to Problem 5

The final velocity is found by

$$v = v_0 + at = 12 \ ^{m}\!/_{s} + \left(2.5 \ ^{m}\!/_{s^2}\right)(15 \text{ s}) = \mathbf{50} \ ^{\mathbf{m}}\!/_{\mathbf{s}}$$

We need one more kinematic equation to relate initial velocity, final velocity, acceleration and displacement. Since it is free of time t, we can call it the *time-independent equation*. It is left to the student to derive it.

$$v^2 = v_0^2 + 2ax$$

When the velocity and acceleration of a body are in the same direction, the velocity of the body increases with time. When the velocity and acceleration of the body are in opposite directions, the velocity of the body decreases with time.

Kinematics and Graphic Relationships

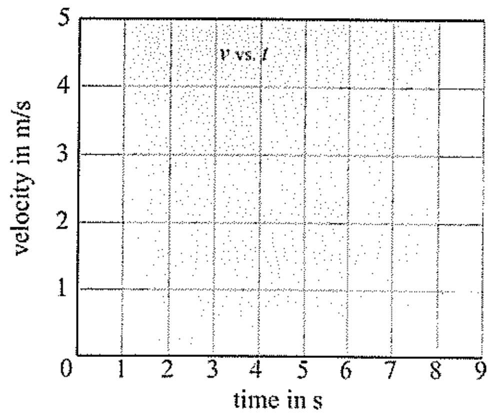
(*College Physics* 9th ed. pages 29–36/10th ed. pages 31–38)

A convenient method of determining the displacement from a velocity vs. time graph is simply to find the area under the curve. Consider the following graph. A particle travels with a constant velocity of $v = 5 \ ^{m}\!/_{s}$. What is the displacement of the particle in the time interval from 1 s to 8 s? The area bound by the time interval is shaded. Its area is that of a rectangle. The area is $A =$ base × heighth =

$(8\text{ s}-1\text{ s})\left(5\ ^{\text{m}}\!/_{\text{s}}\right)=35\text{ m}$. During the time interval between 1 s and 8 s, the particle had a displacement of 35 m.

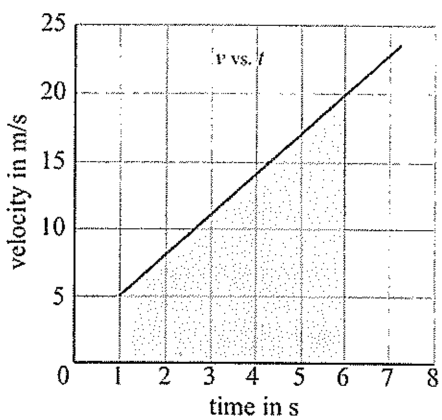
Consider another approach, constant velocity is expressed as $v=\dfrac{x}{t}$ and the displacement is $x=vt=\left(5\ ^{\text{m}}\!/_{\text{s}}\right)(8\text{ s}-1\text{ s})=35\text{ m}$.

The "curve" is a horizontal line and the slope of a horizontal line is zero. The slope of the curve on a velocity vs. time graph is the acceleration. Since the slope is zero, the particle has an acceleration of zero.



Velocity vs. time graph for constant velocity, showing that the displacement is given by the area beneath the curve.

Consider a body having the velocity vs. time graph given below. Find the displacement of the body over the time interval from 1 s to 6 s.



Velocity vs. time graph for uniform acceleration, showing that the displacement is given by the area beneath the curve.

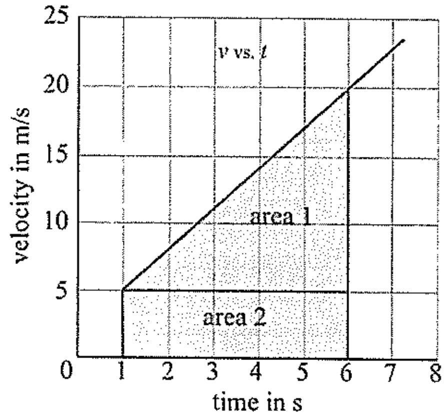
Again, the displacement of the body is the area beneath the velocity vs. time graph. The shaded area under the curve shows the displacement. The shaded area is actually two separate areas, area 1 and area 2. Area 1 is a triangle and its area is

$$A1=\frac{1}{2}\text{base}\times\text{altitude}=\frac{1}{2}(6\text{ s}-1\text{ s})\left(20\ ^{\text{m}}\!/_{\text{s}}-5\ ^{\text{m}}\!/_{\text{s}}\right)=37.5\text{ m}$$

Area 2 is a rectangle and its area is

$$A2=\text{base}\times\text{height}=(6\text{ s}-1\text{ s})\left(5\ ^{\text{m}}\!/_{\text{s}}\right)=25\text{ m}$$

The total area is $A1 + A2 = 37.5$ m + 25 m = 62.5 m. In the 5 s time interval the body is displaced by 62.5 m.



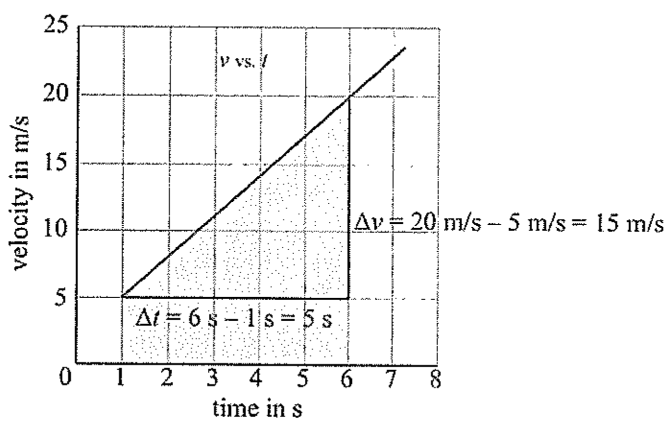
Note that the shape of the curve is a straight-line sloped up to the right. This means that the velocity of the body is changing; it is accelerating. Since it is sloped up to the right, the body undergoes a positive acceleration. Another feature of a velocity vs. time graph is that we can determine acceleration.

We define the slope of a straight line as

$$\text{slope}=\frac{\text{change in the vertical}}{\text{change in the horizontal}}$$

Delta, Δ, is the symbol we use for change. Slope can be defined as

$$m=\text{slope}=\frac{\Delta y}{\Delta x}$$



Note that the acceleration process starts at $t = 1$ s. The velocity at this time, the initial velocity, is $5\ \mathrm{m/s}$.

Since we have a graph with the velocity along the vertical and the time along the horizontal we write

$$m=\frac{\Delta v}{\Delta t}=\frac{v-v_0}{t-t_0}=\frac{20\ \mathrm{m/s}-5\ \mathrm{m/s}}{6\ \mathrm{s}-1\ \mathrm{s}}=\frac{15\ \mathrm{m/s}}{5\ \mathrm{s}}=3\ \mathrm{m/s^2}$$

The body is accelerating with a positive $3\ \mathrm{m/s^2}$.

The displacement of a body undergoing acceleration is defined as

$$x=v_0t+\frac{1}{2}at^2=\left(5\ \mathrm{m/s}\right)(5\ \mathrm{s})+\frac{1}{2}\left(3\ \mathrm{m/s^2}\right)(5\ \mathrm{s})^2=62.5\ \mathrm{m}$$

Graphs are pictures framed with a Cartesian coordinate system. Is a graph worth a thousand words?

THE BASIC EQUATIONS OF KINEMATICS

(*College Physics* 9th ed. pages 36–42/10th ed. pages 38–44)

In our study of how things move we require the following five kinematic equations in dealing with motion along the horizontal.

Average velocity	$\bar{v} = \frac{x}{t} = \frac{v + v_0}{2}$
Acceleration	$a = \frac{v - v_0}{t}$
Final velocity	$v = v_0 + at$
Displacement	$x = v_0 t + \frac{1}{2}at^2$
Time-Independent relationship	$v^2 = v_0^2 + 2ax$

We have looked at one-dimensional motion along the *x*-axis. Next, we look at motion along the vertical, the *y*-axis.

Equations with x as displacement are specifically for motion along the *x*-axis. We use y as displacement along the vertical. To generalize our equations we will use s for displacement. Our generalized kinematic equations are then:

Average velocity	$\bar{v} = \frac{s}{t} = \frac{v + v_0}{2}$
Acceleration	$a = \frac{v - v_0}{t}$
Final velocity	$v = v_0 + at$
Displacement	$s = v_0 t + \frac{1}{2}at^2$
Time-Independent relationship	$v^2 = v_0^2 + 2as$

FREE FALL

(*College Physics* 9th ed. pages 43–47/10th ed. pages 44–49)

Much of our understanding about the behavior of falling bodies was developed by Galileo Galilei (1564–1642). He was the first to show that in the absence of air resistance all bodies, regardless of their size or weight, fall to the Earth with the same acceleration. The Earth behaves as if all its mass were concentrated at a single point at the very center of our planet. And again, we call this point the center of mass, c.m. of the Earth. Due to its mass, the Earth generates a *gravitational field* that extends into deep space. Our concern at the moment is the gravitational field within several kilometers of the surface of the Earth. As a first approximation, we treat the gravitational field as uniform with uniform intensity. Fields are vector quantities that have magnitude, a unit, and a direction. The magnitude and unit of the gravitational field is 9.80 $\mathrm{m/s^2}$. The direction is vertically downward.

Note the unit m/s^2 is the same unit as acceleration. The intensity is an acceleration and we call it the *acceleration due to gravity* and we give it its own special symbol, g.

$$g = \text{acceleration due to gravity} = -9.80\ \frac{\text{m}}{\text{s}^2}$$

The negative sign (–) gives the direction of g, vertically downward toward the center of the Earth.

In the gravitational field of the Earth, neglecting air resistance, all compact bodies fall with the same acceleration, g, the acceleration due to gravity.

VERTICAL DISPLACEMENT AND VELOCITY

(*College Physics* 9th ed. pages 43–47/10th ed. pages 44–49)

In general, the displacement of a body is $s = v_0t + \frac{1}{2}at^2$. Treating vertical motion as along the y-axis and acceleration as g yields the displacement equation for free fall: $y = v_0t - \frac{1}{2}gt^2$. Modifying $v = v_0 + at$ for the velocity of a body in free fall gives $v = v_0 - gt$. Note that the negative sign for g is built into the equations.

In our study of how things move vertically upward or downward, we require the following three free-fall equations.

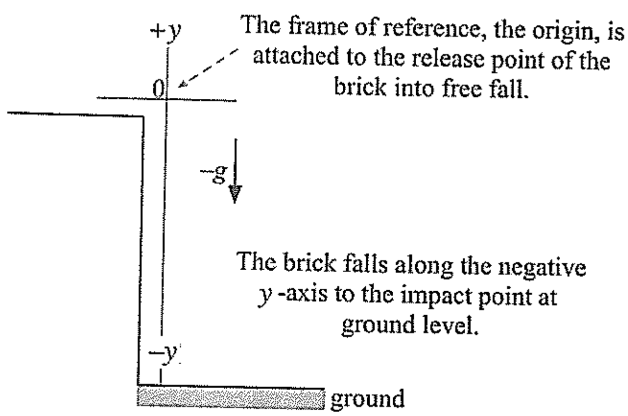
Vertical displacement	$y = v_0t - \frac{1}{2}gt^2$
Final velocity	$v = v_0 - gt$
Time-independent relationship	$v^2 = v_0^2 - 2gy$

SAMPLE PROBLEM 6

A person standing at the edge of a cliff drops a brick from rest. The brick impacts the ground 2.4 seconds later.

(a) How far did the brick fall?

(b) What is the velocity of the brick upon impact with the ground?



Solution to Problem 6

(a) The brick is released from rest making $v_0 = 0$. The distance fallen is $y = v_0 t - \frac{1}{2}gt^2 = (0)t - \frac{1}{2}\left(9.80\ \frac{\text{m}}{\text{s}^2}\right)(2.4\ \text{s})^2 = \mathbf{-28.22\ m}$

The negative sign implies that the brick has fallen vertically downward from its point of release.

(b) Since $v_0 = 0$, the velocity on impact is:

$$v = v_0 - gt = 0 - \left(9.80\ \frac{\text{m}}{\text{s}^2}\right)(2.4\ \text{s}) = \mathbf{-23.52\ {}^{m}\!/\!_{s}}$$

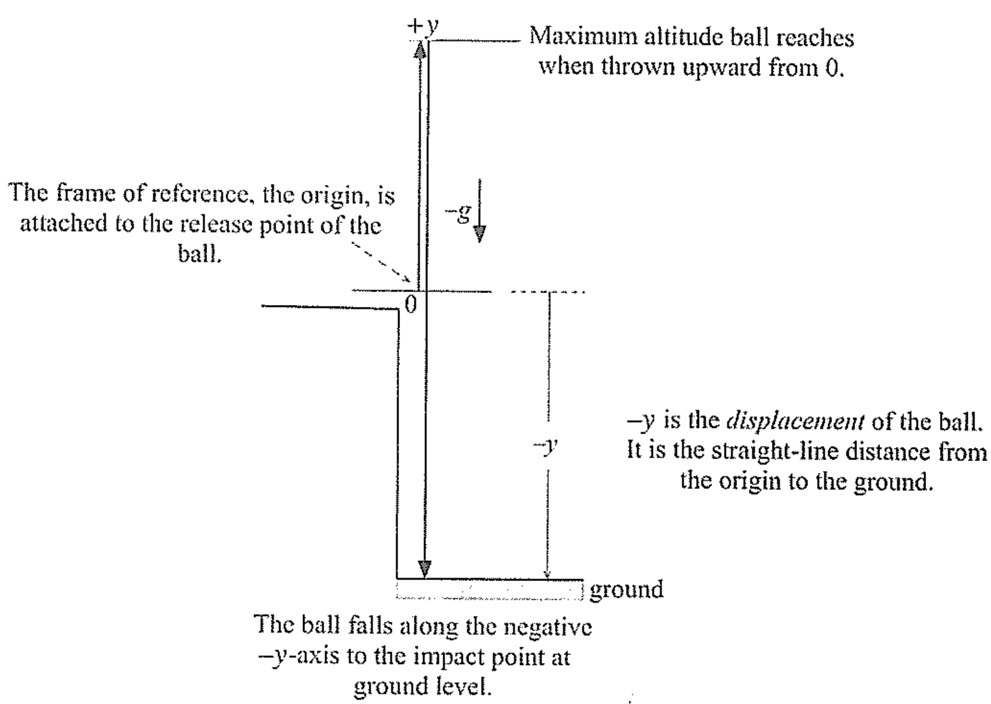
The negative sign means that the brick is falling vertically downward.

Sample Problem 7

A person standing at the edge of a cliff 30.0 m in height reaches out over the edge and throws a ball vertically upward with a velocity of 12.0 $^{m}\!/\!_{s}$. The ball slows as it rises and reaches a point where it stops momentarily. The ball then falls to the ground just missing the edge of the cliff.

(a) How long is the ball in flight?

(b) How long is the ball in flight if it is thrown vertically downward at 12.0 $^{m}\!/\!_{s}$?



SOLUTION TO PROBLEM 7

(a) The ball ends up 30 m below the origin making its displacement $y = -30\text{ m}$. The initial velocity of the ball is $+12.0\ \text{m/s}$ since the ball was thrown vertically upward. Starting with $y = v_0 t - \frac{1}{2}gt^2$ we solve for the time of flight, t. To save time and space we are *not* going to enter the units for each term in this problem. Substituting numbers in,

$$-30 = 12t - \frac{1}{2}(9.8)t^2 = 12t - 4.9t^2.$$

Transposing and rearranging terms gives the $4.9t^2 - 12t - 30 = 0$. Note this is of the form $at^2 + bt + c = 0$, a quadratic equation solvable with the quadratic formula $t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Making substitution into the quadratic formula $t = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(4.9)(-30)}}{2(4.9)}$. There are two solutions, $t = 4.0\text{ s}$ and $t = -1.5\text{ s}$. The negative time is not valid. Did the ball leave the hand of the person throwing it upward 1.5 s before it was thrown? Of course not, the correct answer is $\mathbf{t = 4.0\ s}$.

(b) The ball is thrown downward at $-12.0\ \text{m/s}$. It has a displacement of $y = -30\text{ m}$. The quadratic equation that relates these and the time of flight is $4.9t^2 + 12t - 30 = 0$. The quadratic formula once again gives two roots, $t = -4.0\text{ s}$ and $t = 1.5\text{ s}$. Compare these with the ones in part (a). The correct answer is $\mathbf{t = 1.5\ s}$.

AP Tip

Distances above the origin are positive where distances below the origin are negative. Upward velocities are positive and downward velocities are negative. Acceleration in free fall is downward and negative.

When a body is projected vertically upward, its velocity will rapidly diminish until at some point it comes momentarily to rest and then falls back toward the Earth, acquiring again at the ground the same speed it had upon projection.

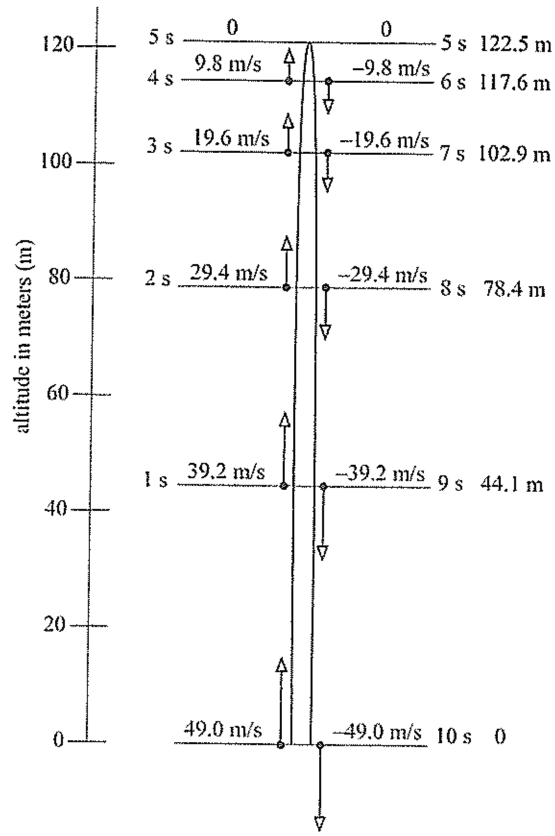
Study the diagram presented below. A ball is projected vertically upward from ground level at $49.0\ \text{m/s}$. Of course we consider the ball to be a compact body and we ignore air resistance. At the end of 1.0 s of flight, it is 44.1 m above the ground traveling upward at $39.2\ \text{m/s}$.

At the end of 2.0 s it has a velocity of $29.4\ \text{m/s}$ and is now 78.4 m above the ground. Notice the ball is slowing with time and altitude. In all cases it has the same acceleration, $g = -9.8\ \text{m/s}^2$.

At the end of 3.0 s it is 102.9 m above the ground traveling at 19.6 $^{m}/_{s}$.

At $t = 4$ s it has slowed to 9.8 $^{m}/_{s}$ and is located at 117.8 m above ground level. At the end of 5.0 s the ball reaches a maximum altitude of 122.5 m and for a tiny instant has stopped. Then it begins to fall, accelerating at g as it does.

Note the symmetry in the diagram.



The time it takes to reach the top of its trajectory is equal to the time taken from there to the ground. This implies that the upward motions are just the same as the downward motions, but in reverse.

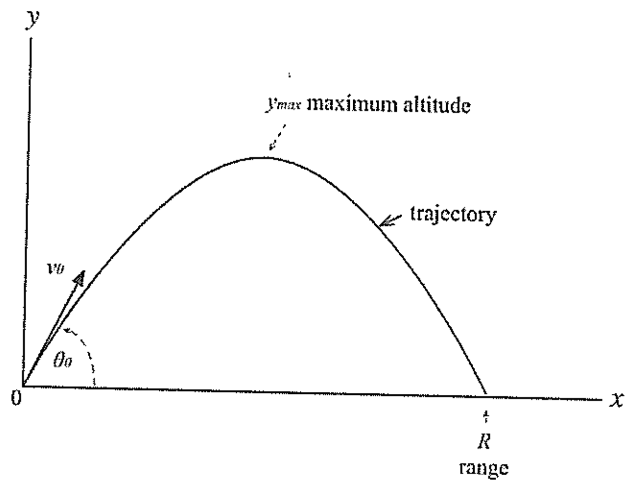
PROJECTILES

(*College Physics* 9th ed. pages 63–71/10th ed. pages 65–73)

A compact body thrown or projected into 2-D space is called a *projectile*. Its path in flight is called its *trajectory*. As a first approximation, we ignore air resistance and consider projectiles as compact bodies that travel short distances horizontally and vertically. Under these conditions the trajectory is considered *parabolic*.

Let v_0 at θ_0 be the initial velocity of the projectile. Remember that velocity is a vector having both magnitude v and direction θ. We let

y_{max} be the *maximum altitude* reached by the projectile and R or X_{max} be the *maximum range* of the projectile. We make t_T the total time of flight of the projectile.



Initial Velocity Components

(*College Physics* 9th ed. pages 63–71/10th ed. pages 65–73)

The initial velocity vector has a set of two components, horizontal v_{0x} and vertical v_{0y}. Components of a vector are mutually perpendicular and form a right triangle with the initial velocity vector. Because of this we can write:

$$\cos\theta_0 = \frac{v_{0x}}{v_0} \text{ and } \sin\theta_0 = \frac{v_{0y}}{v_0}$$

$$v_{0x} = v_0 \cos\theta_0 \quad v_{0y} = v_0 \sin\theta_0$$

Both equations are needed to calculate the horizontal and vertical components of the initial velocity.

Sample Problem 8

A projectile is fired into the air with an initial velocity of $v_0 = 40.0\ \text{m/s}$ at an angle $\theta_0 = 60.0°$. Calculate the initial velocity components.

Solution to Problem 8

We write

$$v_{0x} = v_0 \cos\theta_0 = \left(40.0\ \text{m/s}\right)\cos 60° = \mathbf{20.0\ m/s}$$

$$v_{0y} = v_0 \sin\theta_0 = \left(40.0\ \text{m/s}\right)\sin 60° = \mathbf{34.6\ m/s}$$

INSTANTANEOUS VELOCITY OF A PROJECTILE IN FLIGHT

(*College Physics* 9th ed. pages 63–71/10th ed. pages 65–73)

Ignoring air resistance, the only force acting on the projectile in flight is the force of gravity. The force of gravity is an unbalanced force and causes a vertically downward acceleration, g, where the vertical acceleration, a_y, is $a_y = -g$. There is no horizontal force acting on the projectile and $a_x = 0$.

In general, the kinematic relationship for velocity and acceleration is: $v = v_0 + at$. For the horizontal velocity component, v_x:

$$v_x = v_{0x} + at = v_{0x} + (0)t$$

$$v_x = v_0 \cos\theta_0$$

Note that everywhere along its trajectory, the horizontal component of velocity of the projectile is a constant.

For the vertical velocity component, v_y:

$$v_y = v_{0y} + at = v_{0y} + (-g)t$$

$$v_y = v_0 \sin\theta_0 - gt$$

To find the instantaneous velocity, v, at any point along the trajectory we note that v_x and v_y are mutually perpendicular and that they form a right triangle with v, and:

$$v^2 = v_x^2 + v_y^2$$

Then

$$v = \sqrt{v_x^2 + v_y^2}$$

And the angle of v is found from:

$$\tan\theta = \frac{v_y}{v_x}$$

and

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

AP Tip

To find the instantaneous velocity of the projectile in flight along its trajectory we must to do <u>four</u> things:

First, find v_x from $v_x = v_0 \cos\theta_0$.

Second, determine v_y using $v_y = v_0 \sin\theta_0 - gt$.

Third, find the magnitude of the instantaneous velocity, v, using $v = \sqrt{v_x^2 + v_y^2}$.

Fourth, find the angle, θ, of the instantaneous velocity vector by $\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$.

SAMPLE PROBLEM 9

Considering Sample Problem 8, calculate the instantaneous velocity of the projectile at t = 2.0 s into its flight.

SOLUTION TO PROBLEM 9

To find the instantaneous velocity of the projectile 2.0 s into flight we do four things:

First $v_x = v_0 \cos\theta_0 = \left(40.0\ ^{m}\!/_{s}\right)\cos 60° = 20.0\ ^{m}\!/_{s}$

Second $v_y = v_0 \sin\theta_0 - gt = \left(40.0\ ^{m}\!/_{s}\right)(\sin 60°) - \left(9.8\ ^{m}\!/_{s^2}\right)(2.0\ \text{s}) = 15.0\ ^{m}\!/_{s}$

Third $v = \sqrt{v_x^2 + v_y^2} = \sqrt{\left(20.0\ ^{m}\!/_{s}\right)^2 + \left(15.0\ ^{m}\!/_{s}\right)^2} = 25.0\ ^{m}\!/_{s}$

Last $\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{15.0\ ^{m}\!/_{s}}{20.0\ ^{m}\!/_{s}}\right) = 36.9°$

At t = 2.0 s into its flight, the projectile has a velocity of $v = \mathbf{25.0\ ^{m}\!/_{s}\ @\ 36.9°}$

LOCATING A PROJECTILE ALONG ITS TRAJECTORY

(*College Physics* 9th ed. pages 63–71/10th ed. pages 65–73)

The horizontal velocity component of the projectile in flight is a constant $v_x = v_0 \cos\theta_0$ and the horizontal velocity is also equal to the horizontal displacement, x, divided by the time the projectile has been in flight, t, to that point, or $v_x = \frac{x}{t}$.

The horizontal distance the projectile has traveled from its launch point is

$$v_x = \frac{x}{t} = v_0 \cos\theta_0$$

or

$$x = (v_0 \cos\theta_0)t$$

The vertical distance the projectile has traveled above the plane of launch is

$$y = v_{0y}t - \frac{1}{2}gt^2.$$

From equation (2) $v_{0y} = v_0 \sin\theta_0$. Substitution yields

$$y = (v_0 \sin\theta_0)t - \frac{1}{2}gt^2$$

AP Tip

To locate the position of a projectile along its trajectory we do two things:

First, determine its distance down range using $x = (v_0 \cos\theta_0)t$.

Second, find its altitude using $y = (v_0 \sin\theta_0)t - \frac{1}{2}gt^2$.

Sample Problem 10

Calculate the position of the projectile of Sample Problem 8 at t = 2.0 s.

Solution to Problem 10

To locate the projectile 2.0 s into flight we do two things:

First:

$$x = (v_0 \cos\theta_0)t = \left(40.0\ \text{m/s}\right)(\cos 60°)(2.0\ \text{s}) = 40.0\ \text{m}$$

Second:

$$\begin{aligned} y &= (v_0 \sin\theta_0)t - \frac{1}{2}gt^2 \\ &= \left(40.0\ \text{m/s}\right)(\sin 60°)(2.0\ \text{s}) - \frac{1}{2}\left(9.8\ \text{m/s}^2\right)(2.0\ \text{s})^2 \\ &= 49.7\ \text{m} \end{aligned}$$

At t = 2.0 s into its flight, the projectile is **40.0 m** down range and is **49.7 m** above the ground.

Notice that both of the above location equations are *parametric* equations with time, t, as the *parameter*. They are both functions of time: $x = f(t)$ and $y = h(t)$. If we wish to express $y = h(x)$ we need to solve $x = (v_0 \cos\theta_0)t$ for t and

$$t = \frac{x}{(v_0 \cos\theta_0)}$$

Substituting for t into $y = (v_0 \sin\theta_0)t - \frac{1}{2}gt^2$:

$$y = (v_0 \sin\theta_0)\left(\frac{x}{v_0 \cos\theta_0}\right) - \frac{g}{2}\left(\frac{x}{v_0 \cos\theta_0}\right)^2$$

In the first term v_0's divide out and $\frac{\sin\theta_0}{\cos\theta_0} = \tan\theta_0$, then we write

$$y = x\tan\theta_0 - \frac{gx^2}{2v_0^2 \cos^2\theta_0}$$

Note that g, v_0 and θ_0 are constants putting the equation into the form

$$y = bx - ax^2$$

which is the equation of an inverted parabola with the projection point being the origin.

SAMPLE PROBLEM 11

Considering the projectile of Sample Problem 8, how far above the ground is the projectile when it has traveled 20.0 m down range?

SOLUTION TO PROBLEM 11

The altitude of the projectile is

$$y = x\tan\theta_0 - \frac{gx^2}{2v_0^2\cos^2\theta_0} = (20.0\text{ m})(\tan 60°) - \frac{\left(9.8\ \text{m/s}^2\right)(20.0\text{ m})^2}{2\left(40.0\ \text{m/s}\right)^2(\cos 60°)^2}$$

$$y = 34.64\ \text{m} - \frac{(9.8)(400)\ \text{m}}{2(1600)(0.50)^2} = (34.64 - 4.90)\text{m} = \mathbf{29.7\ m}$$

The projectile will be 29.7 m above the ground when it has traveled 20.0 m down range.

TOTAL TIME OF FLIGHT OF A PROJECTILE

(*College Physics* 9th ed. pages 63–71/10th ed. pages 65–73)

Writing $y = (v_0\sin\theta_0)t - \frac{1}{2}gt^2$, the projectile returns to the ground, $y = 0$, at t_T, the total time of flight.

$$0 = (v_0\sin\theta_0)t_T - \frac{1}{2}gt_T^2$$

Solving for t_T: $gt^2 = 2(v_0\sin\theta_0)t_T$. A t_T on each side divides out, leaving

$$t_T = \frac{2v_0\sin\theta_0}{g}$$

THE RANGE OF A PROJECTILE

(*College Physics* 9th ed. pages 63–71/10th ed. pages 65–73)

The horizontal distance down range a projectile will travel is found from $x = (v_0\cos\theta_0)t$. The projectile will reach maximum distance down range at t_T. Substitution yields:

$$R = x_{\text{max}} = (v_0\cos\theta_0)t_T = (v_0\cos\theta_0)\left(\frac{2v_0\sin\theta_0}{g}\right)$$

$$R = v_0^2\left(\frac{2\sin\theta_0\cos\theta_0}{g}\right)$$

To simplify the equation we make use of the trig double angle identity, $2\sin\theta_0\cos\theta_0 = \sin 2\theta_0$.

$$R = \frac{v_0^2 \sin 2\theta_0}{g}$$

Because of the properties of trig functions, sets of complementary angles will give the same range. Projectiles fired at shallow angles spend less time in flight than ones fired at steeper angles. An angle of 45° will give maximum range.

Maximum Altitude of a Projectile

(*College Physics* 9th ed. pages 63–71/10th ed. pages 65–73)

By symmetry, it takes as long for a projectile to reach maximum altitude, y_{max}, as it does to return to the ground at $y = 0$ and $X = R$. The time, $t_{T/2}$, to reach y_{max} is half of the total time of flight or

$$t_{T/2} = \frac{v_0 \sin\theta_0}{g}$$

To find y_{max}, substitute into $y = (v_0 \sin\theta_0)t - \frac{1}{2}gt^2$.

$$y_{max} = (v_0 \sin\theta_0)t_{T/2} - \frac{1}{2}gt_{T/2}^2 = (v_0 \sin\theta_0)\left(\frac{v_0 \sin\theta_0}{g}\right) - \frac{1}{2}g\left(\frac{v_0 \sin\theta_0}{g}\right)^2$$

$$y_{max} = \frac{v_0^2 \sin^2\theta_0}{2g}$$

Sample Problem 12

The projectile of Sample Problem 8 has an initial velocity of $v_0 = 40.0\ \mathrm{m/s}$ at an angle $\theta_0 = 60.0°$. Determine

(a) the total time the projectile spends in flight.

(b) the range of the projectile.

(c) the maximum altitude reached by the projectile.

Solution to Problem 12

(a) total time of flight is

$$t_T = \frac{2v_0 \sin\theta_0}{g} = \frac{2\left(40.0\ \mathrm{m/s}\right)(\sin 60°)}{\left(9.8\ \mathrm{m/s^2}\right)} = \mathbf{7.1\ s}$$

(b) range is given by

$$R = \frac{v_0^2 \sin 2\theta_0}{g} = \frac{\left(40.0\ \mathrm{m/s}\right)^2 \sin(2\times 60°)}{9.8\ \mathrm{m/s^2}} = \frac{(1600)\sin 120°}{9.8}\,\mathrm{m} = 141.4\ \mathrm{m}$$

(c) maximum altitude is

$$y_{max} = \frac{v_0^2 \sin^2 \theta_0}{2g} = \frac{\left(40.0\ \mathrm{m/s}\right)^2 (\sin 60°)^2}{2\left(9.8\ \mathrm{m/s^2}\right)} = 61.2\ \mathrm{m}$$

Motion: Student Objectives for the AP Exam

- You should be able to define displacement, velocity, time interval and acceleration.
- You should be able to calculate velocity and acceleration from time intervals and displacement.
- You should be able to explain the subsequent motion of a particle that, at one instant, has zero velocity but constant acceleration.
- You should be able to discuss the subsequent motion of a particle that has negative acceleration.
- You should be able to discuss the motion of a particle that, at one instant, has negative velocity but a positive acceleration.
- You should be able to discuss the properties of a velocity vs. time graph.
- You should be able to analyze the motion of a body in free fall.
- You should be able to analyze the motion of a projectile launched into two-dimensional space.

Multiple-Choice Questions

1. A body moves in the *x-y* plane with some initial velocity v_0, a short time later it has a velocity v_f. Which situation is impossible?
   (A) Velocity and acceleration vectors are parallel.
   (B) Velocity and acceleration vectors are anti-parallel.
   (C) Velocity and acceleration vectors are perpendicular.
   (D) Velocity and acceleration are both constant (nonzero).